LILAVATI

LILAVATI

Translated by

COLEBROOKE

WITH NOTES

By

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PREFACE.

THIS little book is an edition of Colebrooke's translation of the *Lílávatí*, a standard work on Hindu mathematics, written by Bháskaráchároyya, a celebrated mathematician and astronomer who lived in the twelfth century of the Christian era.[1] The work forms the first part of a larger work of the author called the *Siddhánta-siromani*. This part is called by the author, *Pátíganita* or Arithmetic; but this name has not been properly given. For, the work, besides dealing with subjects which lie within the province of Arithmetic, also treats of subjects which properly belong to Algebra and Geometry. It comprises the four simple rules, extraction of the square root and the cube root, vulgar fractions, Rule of Three, interest, alligation, problems producing simple and quadratic equations, arithmetical and geometrical progressions, permutations and combinations, indeterminate equations of the first degree, several properties of triangles and quadrilaterals, areas of circles, volumes of spheres, cones and pyramids, solid content of excavations, and several other matters. Some of the problems solved evince a great deal of progress in algebraical investigations. The author does not state the reasons for the various rules given by him. I have tried to supply the reasons as simply and shortly as they occurred to me; but still in some cases neater and shorter demonstrations may possibly be given. The explanations given have been printed in small type and enclosed within square brackets. It is thus hoped that the present edition will prove useful and interesting not

[1] This date is ascertained from the fact that Bháskara himself informs us in a passage of his *Siddhánta-siromaui*, that he was born in the year 1036 of the *Saka* era, and that he completed his great work when he was 36 years old. This gives 1150 A.D., as the date of the completion of the *Siddhánta-siromani*. See the *Goládhyáya* of the *Siddhánta-siromani*, Wilkinson's translation, XIII, 58.

only to the scholar and the antiquarian, but also to the student of modern algebra.

In his foot-notes, Colebrooke has given translations of extracts from the leading commentaries on the *Lílávatí*. These are:— (1) The commentary of Gangádhara, written about 1420 A.D.: (2) that of Súryadása, called *Ganitámrita*, written in 1538 A.D., containing a clear interpretation of the text, with concise explanations of the rules: (3) that of Ganesa, called *Buddhivilásiní*, the best of all the commentaries, written in 1545 A.D., comprising a copious exposition of the text, with demonstrations of the rules: (4) the gloss of Ranganátha on the *Vásaná*, or Bháskara's demonstratory annotations of the *Siddhánta-siromani*, written towards the beginning of the seventeenth century A.D.: (5) the *Manoranjana*, written by Rám Krishna Deva, of uncertain date: and (6) the *Ganitakaumudí*, which has not been recovered, but is known from the quotations cited from it by Súryadása and Ranganátha. Some of the translated extracts contain expositions of the rules and of technical terms, and some contain demonstrations of the rules in a few cases. Of these demonstrations which are given chiefly by Ganesa and Súryadása, those which are satisfactory and instructive, have been retained in the present edition; whilst others which are obscure and unsatisfactory, have been omitted. For convenience of reference, the *Lílávatí* in Sanskrit is printed at the end, with divisions into chapters and sections corresponding to those made in the translation. No such divisions were made by Bháskara.

H. C. B.

NARIKELDANGA, CALCUTTA,

CONTENTS

LÍLÁVATÍ.

CHAPTER I.

INTRODUCTION.

1. Having bowed to the deity, whose head is like an elephant's[1]; whose feet are adored by gods; who, when called to mind, relieves his votaries from embarrassment; and bestows happiness on his worshippers; I propound this easy process of computation,[2] delightful by its elegance,[3] perspicuous with words concise, soft and correct, and pleasing to the learned.

DEFINITIONS OF TECHNICAL TERMS.

(*Money by tale.*)

2. Twice ten cowry shells[4] are a *kákiní;* four of these are a *pana;* sixteen of which must be here considered as a *dramma;* and in like manner, a *nishka,* as consisting of sixteen of these.

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[1] Ganesa, represented with an elephant's head and human body.

[2] *Pátíganita; pátí, paripátí,* or *vyaktaganita,* arithmetic.

[3] *Líláwatí,* delightful: an allusion to the title of the book. See notes on §§ 13 and 277.

[4] Cypræa moneta. Sans., *Varátaka, kapardí.* Hindi, *Kaurí.*

(*Weights.*)

3. A *gunja*[1] (or seed of *Abrus*) is reckoned equal to two barley-corns ; a *valla*, to three *gunjas;* and eight of these are a *dharana;* two of which make a *gadyánaka.* In like manner one *dhataka* is composed of fourteen *vallas*.

4. Half ten *gunjas* are called a *másha*, by such as are conversant with the use of the balance ; a *karsha* contains sixteen of what are termed *máshas;* a *pala*, four *karshas*. A *karsha* of gold is named *suvarna*.

(*Measures.*)

5—6. Eight breadths of a barley-corn[2] are here a finger; four times six fingers, a cubit[3]; four cubits, a staff[4]; and a *krosa* contains two thousand of these; and a *yojana*, four *krosas*.

So a bambu pole consists of ten cubits; and a field (or plane figure) bounded by four sides, measuring twenty bambu poles, is a *nivartana*.[5]

7. A cube,[6] which in length, breadth and thickness measures a cubit, is termed a solid cubit: and, in the meting of corn and the like, a measure, which contains

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[1] A seed of Abrus precatorius, black or red ; the one called *Krishnala*, the other *raktiká*, *ratti*, or *rattiká*.

[2] Eight barley-corns (*yava*) by breadth, or three grains of rice by length are equal to one finger (*angula*). —Gan.

[3] According to Ganesa, the cubit (*hasta*) means the practical cubit employed by artisans and called *gaj*. It is longer than the ordinary cubit of 18 inches.

[4] *Danda*, a staff: directed to be cut nearly of man's height. (Manu, II. 46.)

[5] A superficial measure containing 400 square poles.—Súr.

[6] *Dwádasásra*, lit. dodecagon, but meaning a parallelopiped; the term *asra*, corner or angle, being here applied to the edge or line of incidence of two planes.

a solid cubit, is a *khárí* of *Magadha*[1] as it is denominated in science.

8. A *drona* is the sixteenth part of a *khárí;* an *ádhaka* is a quarter of a *drona;* a *prastha* is a fourth part of an *ádhaka;* and a *kudaba* is by the ancients termed a quarter of a *prastha.*[2]

The rest of the axioms, relative to time[3] and so forth, are familiarly known.

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[1] The country situated on the *Sonebhadrá* river.—Gan. It is South Behar.

[2] Another stanza occurs here in one copy of the text. It is explained in the *Manoranjana,* and by Gangádhara, but not by Ganesa and Súryadása. It is therefore to be rejected as spurious and interpolated. It is as follows:— "The *sera* is here reckoned at twice seven *tankas,* each equal to three-fourths of a *gadyánaka:* and a *mana,* at forty *seras.* The name is in use among the *Turushkas,* for a weight of corn and like articles." See notes on §§ 97 and 233.

[3] The author has himself explained the measures of time in his *Siddhánta-siromani.* [See the *Goládhyáya,* Wilkinson's translation, IV. 5—12.—Ed.]

CHAPTER II.

SECTION I.

INVOCATION.[1]

9. Salutation to Ganesa, resplendent as a blue and spotless lotus; and delighting in the tremulous motion of the dark serpent, which is perpetually twining within his throat.

NUMERATION.

10—11. Names of the places of figures have been assigned for practical use by ancient writers,[2] increasing regularly[3] in decuple proportion: namely, unit, ten, hundred, thousand, myriad, hundred thousands, million, ten millions, hundred millions, thousand millions, ten thousand millions, hundred thousand millions, billion,

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[1] A reason of this second introductory stanza is, that the foregoing definitions of terms are not properly a part of the treatise itself; none such having been premised by Árya Bhatta and other ancient authors in their treatises on arithmetic.—Gan. and *Mano.*

[2] According to the Hindus, numeration is of divine origin; the invention of nine figures (*anka*), with the device of places to make them suffice for all numbers, being ascribed to the beneficent Creator of the Universe, in Bháskara's *Vásaná* and its gloss; and in Krishna's Commentary on the *Vija-ganita*. Here nine figures are specified; the place, when none belongs to it, being shown by a blank (*súnya*), which, to obviate mistake, is denoted by a dot or small circle.

[3] From the right, where the first and lowest number is placed, towards the left hand.—Gan.

ten billions, hundred billions, thousand billions, ten thousand billions, hundred thousand billions.[1]

SECTION II.

EIGHT OPERATIONS[2] OF ARITHMETIC.

12. Rule of addition and subtraction[3]: half a stanza.

The sum of the figures according to their places is to be taken in the direct or inverse order[4]: or (in the case of subtraction) their difference.

[The rule as exemplified in the *Manoranjana* is more cumbrous than the ordinary rule.]

13. Example. Dear intelligent Lílávatí,[5] if thou be skilled in addition and subtraction, tell me the sum of two, five, thirty-two, a hundred and ninety-three, eighteen, ten, and a hundred, added together; and the remainder, when their sum is subtracted from ten thousand.

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[1] Sans. *eka, dasa, sata, sahasra, ayuta, laksha, prayuta, koti, arbuda, abja* or *padma, kharva, nikharva, mahápadma, sanku, jaladhi* or *samudra, antya, madhya, parárdha.*

A passage of the *Veda*, which is cited by Súryadása, contains the places of figures:—"Be these the milch kine before me, one, ten, a hundred, a thousand, ten thousand, a hundred thousand, a million, ... Be these milch kine my guides in this world."

[2] *Parikarmáshtaka*, eight operations, or modes of process; logistics or algorism.

[3] *Sankalana, sankalita, misrana, yuti, yoga, &c.*, summation, addition. *Vyavakalana, vyavakalita, sodhana, patana, &c.*, subtraction. *Antara*, difference, remainder.

[4] From the first on the right, towards the left; or from the last on the left, towards the right.—Gang.

[5] Seemingly the name of a female to whom instruction is addressed. But the term is interpreted in some of the commentaries, consistently with its etymology, 'charming.'—See §§ 1 and 277.

Statement : 2, 5, 32, 193, 18, 10, 100.

Result of the addition[1] : 360.

Statement for subtraction : 10000, 360.

Result of the subtraction : 9640.

14—15. Rule of multiplication[2]: two and a half stanzas.

Multiply the last[3] figure of the multiplicand by the multiplicator, and next the penult, and then the rest, by the same repeated. Or let the multiplicand be repeated under the several parts of the multiplicator, and be multiplied by those parts : and the products be added together. Or the multiplier being divided by any number which is an aliquot part of it, let the multiplicand be multiplied by that number, and then by the quotient, the result is the product. These are two methods of subdivision by form. Or multiply separately by the places of figures, and add the products together. Or multiply by the multiplicator diminished or increased by a quantity arbitrarily assumed; adding or subtracting the product of the multiplicand taken into the assumed quantity.

[The author gives here six methods. The first method is the ordinary one, and includes the *tatstha* of the older authors, which is worked by repeating or moving the multiplier over or under every digit of the multiplicand, and which, according to Ganesa's

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[1] Mode of working addition as shown in the *Manoranjana* :

Sum of the units, 2,5,2,3,8,0,0,	... ...	2 0
Sum of the tens, 3,9,1,1,0,	... ...	1 4
Sum of the hundreds, 1,0,0,1,	... ...	2
Sum of the sums	... ...	3 6 0

[2] *Gunana*, *abhyása*; also *hanana* and any term implying a tendency to destroy. It is denominated *pratyutpanna* by Brahmagupta and by Srídhara. *Gunya*, multiplicand. *Gunaka*, multiplicator. *Ghàta*, product.

[3] The digit standing last towards the left.

explanation, proceeds obliquely, joining products along compartments. The second is tedious, following from the formula, $a(b+c)=ab+ac$. The third is multiplication by factors. The fourth is practically the same as the first, and the fifth, the same as the second. The sixth follows from the formula, $a(b-c)=ab-ac$.]

16. Example. Beautiful and dear Lílávatí, whose eyes are like a fawn's! tell me the numbers resulting from one hundred and thirty-five, taken into twelve, if thou be skilled in multiplication by whole or by parts, whether by subdivision of form or separation of digits.[1] Tell me, auspicious woman, the quotient of the product divided by the same multiplier.

Statement: Multiplicand 135. Multiplicator 12.

Product (multiplying the digits of the multiplicand successively by the multiplicator) 1620.

Or, subdividing the multiplicator into parts, as 8 and 4; and severally multiplying the multiplicand by them; adding the products together, the result is the same, 1620.

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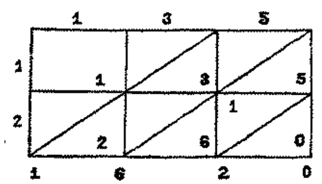
[1] The adjoined scheme of the process of multiplication is exhibited in Ganesa's commentary.



According to the *tatstha* method, the process will stand thus:—

12	12	12	or	135	135
1	3	5		1	2
12		60			270
	36				135
	1620				1620

Or, the multiplicator 12 being divided by 3, the quotient is 4 ; by which, and by 3, successively multiplying the multiplicand, the last product is the same, 1620.

Or, taking the digits as parts, *viz.*, 1 and 2 ; the multiplicand being multiplied by them severally, and the products added together, according to the places of figures, the result is the same, 1620.

Or, the multiplicand being multiplied by the multiplicator less 2, *viz.*, 10, and added to twice the multiplicand, the result is the same, 1620.

Or, the multiplicand being multiplied by the multiplicator increased by 8, *viz.*, 20, and eight times the multiplicand being subtracted, the result is the same, 1620.

17. Rule of division[1]: one stanza. That number, by which the divisor being multiplied balances the last digit of the dividend (and so on[2]), is the quotient in division: or, if practicable, first abridge[3] both the divisor and the dividend by an equal number, and proceed to division.

Example. Statement of the number produced by multiplication in the foregoing example, and of its multiplicator, for a dividend, 1620, and a divisor, 12.

Quotient 135 ; the same with the original multiplicand.[4]

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[1] *Bhága-hára*, *bhájana*, *harana*, *chhedana*, division. *Bhájya*, dividend. *Bhájaka*, *hara*, divisor. *Labdhi*, quotient.

[2] Repeating the divisor for every digit, like the multiplier in multiplication.—Gang.

[3] *Apavartya*, abridging See note on § 249.

[4] The process of long division is exhibited in the *Manoranjana*, thus: The highest places of the proposed dividend, 16, being divided by 12. the quotient is 1 ; and 4 over. Then 42 becomes the highest remaining number, which divided by 12 gives the quotient 3, to be placed in a line with the

Or both the dividend and the divisor, being reduced to least terms by the common measure 3, are 540 and 4; or by the common measure 4, they become 405 and 3. Dividing by the respective reduced divisors, the quotient is the same, 135.

[The first part of the rule is vague and incomplete, although it is practically the same as the ordinary rule, as will be evident from the foot-note 4, p. 8. The second part follows from the identity $ab \div ac = b \div c$.]

18—19. Rule for the square[1] of a quantity: two stanzas.

The multiplication of two like numbers together is the square. The square of the last[2] digit is to be placed over it; and the rest of the digits, doubled and multiplied by that last, to be placed above them respectively; then repeating the number, except the last digit, again (perform the like operation). Or twice the product of two parts, added to the sum of the squares of the parts, is the square (of the whole number.)[3] Or the product of the sum and difference of the number and an assumed quantity, added to the square of the assumed quantity, is the square.[4]

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preceding quotient 1: thus 13. Remainder 60, which divided by 12 gives 5: and this being carried to the same line as before, the entire quotient is exhibited: *viz.*, 135.

[1] *Varga, kriti*, a square number.

[2] The process may begin with the first digit, as intimated by the author in § 24.

[3] The proposed quantity may be divided into three parts, instead of two; and the products of the first and second, first and third, and second and third, being added together and doubled, and added to the sum of the squares of the parts, the total is the square sought.—Gan.

[4] Another method is hinted in the author's note on this passage; consisting in adding together the product of the proposed quantity by any assumed one, and its product by the proposed less the assumed one.—Rang. [This follows from the identity, $ab + a(a-b) = a^2$.—Ed.]

["The square of the last digit, &c." The translation here is slightly incorrect. It should run thus :—"The square of the last digit, and the rest of the digits doubled and multiplied by that last, are to be placed one above the other (regard being had to the local values); then repeating, &c." It will then appear that the first two methods are really the same, and are based on the formula, $(a+b+c)^2=a^2+2a\ (b+c)+b^2+2bc+c^2$. The working of the first method is this :—Suppose we have to find the square of 297.

Then

7^2	=	49	or, 2^2	=	4
$7\times2\times29$	=	406	$2\times2\times97$	=	388
9^2	=	81	9^2	=	81
$9\times2\times2$	=	36	$9\times2\times7$	=	126
2^2	=	4	7^2	=	49
$\therefore(297)^2$	=	88209 ;	$\therefore(297)^2$	=	88209

The ciphers are omitted for simplicity. The third method follows from the identity, $(a+b)(a-b)+b^2=a^2$, a being the proposed, and b, the assumed quantity.]

20. Example. Tell me, dear woman, the squares of nine, of fourteen, of three hundred less three, and of ten thousand and five, if thou know the method of computing the square.

Statement : 9, 14, 297, 10005.

Proceeding as directed, the squares are found : 81, 196, 88209, 100100025. Or, put 4 and 5, parts of 9. Their product doubled 40, added to the sum of their squares 41, makes 81. So, taking 10 and 4, parts of 14, their product 40, being doubled, is 80 ; which, added to 116, the sum of the squares 100 and 16, makes the entire square, 196.

Or, putting 6 and 8, their product 48, doubled, is 96 ; which, added to the sum of the squares 36 and 64, *viz.*, 100, makes the same 196.

Again, 297, diminished by 3, is 294 ; and, in another place, increased by the same, is 300. The product of these is 88200 ; to which adding the square of 3, *viz.*, 9, the sum is as before the square, 88209.

21. Rule for the square root[1] : one stanza.

Having deducted from the last of the odd digits[2] the square number, double its root ; and by that dividing the subsequent even digit, and subtracting the square of the quotient from the next uneven place, note in a line (with the preceding double number) the double of the quotient. Divide by the (number as noted in a) line the next even place, and deduct the square of the quotient from the following uneven one, and note the double of the quotient in the line. Repeat the process (until the digits be exhausted). Half the (number noted in the) line is the root.

[The rule is practically the same as the ordinary one for the extraction of the square root. Only it is a little more cumbrous, as will appear from the two processes placed side by side :—

```
           1__1__1
           8 8 2 0 9 ( 4                      . . .
           4         ( 18                     8 8 2 0 9 ( 297
           ---------                          4
           482         ------                 ---------
  4 × 9 =  36          58               49 | 482
           ------      14                  | 441
           122         ------              | ---------
   9² =    81          594             587 | 4109
           ------      ½ of 594            | 4109
            4109       = 297,              | ---------
 7 × 58 =   406        the root required.
           ------
              49
   7² =       49
           ------
```

[1] *Varga múla*, root of the square; *múla*, *pada*, are synonyms of root.

[2] Every uneven place is to be marked by a vertical line, and the intermediate even digits by a horizontal line. But, if the last place be even, it is joined with the contiguous odd digit. Example, $\overset{|}{8}\overset{-}{8}\overset{|}{2}\overset{-}{0}\overset{|}{9}$

Thus we see that instead of directing the subtraction of 9×49 at once from 482, the rule directs first the subtraction of 9×40, and then from the remainder the subtraction of 9×9 or 9^2. And similarly for the next step. The process shown on the left-hand side is the same as that explained in the *Manoranjana.*]

22. Example. Tell me, dear woman, the root of four, and of nine, and those of the squares before found, if thy knowledge extend to this calculation.

Statement: 4, 9, 81, 196, 88209, 100100025. The roots are 2, 3, 9, 14, 297, 10005.

23—25. Rule for the cube[1]: three stanzas.

The continued multiplication of three like quantities is a cube. The cube of the last (digit) is to be set down; and next the square of the last multiplied by three times the first; and then the square of the first taken into the last and tripled; and lastly, the cube of the first: all these, added together according to their places, make the cube. The proposed quantity (consisting of more than two digits) is distributed into two portions, one of which is then taken for the last (and the other for the first); and in like manner repeatedly (if there be occasion.)[2] Or the same process may be begun from the first place of figures, either for finding the cube or the square. Or three times the proposed number, multiplied by its two parts, added to the sum of the cubes of those parts, give the cube. Or the

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[1] *Ghana*, a cube; lit., solid.

[2] The subdivision is continued until it comes to single digits. Ganesa confines it to the places of figures (*sthána-vibhága*), not allowing the portioning of the number (*rúpa-vibhága*); because the addition is to be made according to the places.

square root of the proposed number being cubed, that multiplied by itself, is the cube of the proposed square.[1]

[The different methods follow from the following formulæ :—

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b).$$

$$\left\{(\sqrt{a^2})^3\right\}^2 = (a^2)^3 .$$]

26. Example. Tell me, dear woman, the cube of nine, the cube of the cube of three, and the cube of the cube of five, and the cube roots of these cubes, if thy knowledge be great in the computation of cubes.

Statement : 9, 27, 125.

The cubes in the same order are, 729, 19683, 1953125.[2]

The proposed number being 9, and its parts 4 and 5, 9 multiplied by them and by 3 is 540 ; which, added to the sum of the cubes 64 and 125, *viz.*, 189, makes

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[1] This carries an allusion to the raising of quantities to higher powers than the cube. Ganesa specifies some of them. Thus the fourth power of a number is called *varga-varga* ; the sixth power, *varga-ghana* or *ghana-varga* ; the eighth power, *varga-varga-varga* ; the ninth power, *ghana-ghana*; the fifth power, *varga-ghana-ghāta* ; and the seventh power, *varga-varga-ghana-ghāta*.

[2] The following process of finding the cube of 125 is given in the *Manoranjana*. The proposed number 125 is distributed into two parts 12 and 5 ; and the first of these again into two parts 1 and 2 :

Then, 1 cubed is	1
1, square of 1, tripled and multiplied by 2, is ...	6
4, square of 2, tripled and multiplied by 1, is ...	12
2 cubed is	8
	1728
Now, 12 cubed as above is	1728
144, square of 12, tripled and multiplied by 5, is ...	2160
25, square of 5, tripled and multiplied by 12, is ...	900
5 cubed is	125
Thus, 125 cubed is	1953125

the cube of 9, *viz.*, 729. The entire number being 27, its parts are 20 and 7 ; by which the number being successively multiplied, and then tripled, is 11340 ; and this added to the sum of the cubes of the parts, 8343, makes the cube 19683.

The proposed number being a square as 4, its root 2 cubed is 8. This taken into itself gives 64, the cube of 4. So 9 being proposed, its square root 3, cubed, is 27 ; the square of which, 729, is the cube of 9. In short, the square of the cube is the same as the cube of the square.

["In short, the square, &c." The translation should be, "in short, the cube of a square number is the same as the square of the cube of the square root of the number." This follows from the third formula given in the preceding article.]

27—28. Rule for the cube root[1]: two stanzas.

The first (digit) is a cube's place ; and the two next, uncubic ; and again, the rest in like manner. From the last cubic place take the (nearest) cube, and set down its root apart. By thrice the square of that root divide the next (or uncubic) place of figures, and note the quotient in a line (with the quantity before found). Deduct its square taken into thrice the last (term), from the next (digit) ; and its cube from the succeeding one. Thus the line (in which the result is reserved) is the root of the cube. The operation is repeated (as necessary).

Example. Statement of the foregoing cubes for extraction of the root : 729, 19683, 1953125.

The cube roots respectively are 9, 27, 125.

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[1] *Ghana-múla*, root of the cube.

[The rule is more cumbrous than the ordinary one, as will appear below :—

```
             1‿ ‿1‿ ‿1
             1953125 ( 1
        1³ = 1          2
             ───────   ────
              953125   12
   3 × 1² = 3             5
      3 × 2 = 6          ───
             ───────     125
              353125
   3 × 2² × 1 = 12
             ───────
              233125
        2³ =   8
             ───────
              225125
3 × 12² = 432
     5 × 432 = 2160
             ───────
                9125
   5² × 12 × 3 = 900
             ───────
                 125
          5³ =   125
             ───────
```

```
                   ˙  ˙  ˙
                   1953125 ( 125
                   1
   3 × 10² = 300 |953
 3 × 10 × 2 = 60 |
         2² =  4 |
           ───── |
             364 |728
                 |225125
  3 × 120² = 43200|
3 × 120 × 5 = 1800|
          5² =  25|
           ───────|
            45025 |225125
```

[The process shown on the left-hand side is the same as that explained in the *Manoranjana*. The ciphers are omitted for simplicity.]

SECTION III.

FRACTIONS.[1]

FOUR RULES FOR THE ASSIMILATION OR REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR.[2]

29. Rule for the simple reduction of fractions[3]: one

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[1] *Bhinna*, a fraction; lit., a divided quantity, or one obtained by division. —Gan. An incomplete quantity or non-integer (*apúrna*).—Gang.

[2] *Bhága-játi-chatushtaya, játi-chatushtaya*, or four modes of assimilation or process for reducing to a common denominator, fractions having dissimilar denominators, preliminary to addition and subtraction of fractions.

[3] *Bhága-játi* or *ansa-savarnana*, assimilation of fractions, reducing them to uniformity.

stanza. The numerator and denominator[1] being multiplied reciprocally by the denominators of the two quantities,[2] they are thus reduced to the same denominator. Or both numerator and denominator may be multiplied by the intelligent calculator into the reciprocal denominators abridged by a common measure.

[This is the ordinary rule for reducing fractions to their least common denominator. The first part of the rule is meant for fractions whose denominators are prime to each other.]

30. Example. Tell me the fractions reduced to a common denominator which answer to three and a fifth, and one-third, proposed for addition; and those which correspond to a sixty-third and a fourteenth offered for subtraction.

Statement[3]: $\frac{3}{1}$, $\frac{1}{5}$, $\frac{1}{3}$.

Reduced to a common denominator, $\frac{45}{15}$, $\frac{3}{15}$, $\frac{5}{15}$. Sum $\frac{53}{15}$.

Statement of the second example: $\frac{1}{63}$, $\frac{1}{14}$.

The denominators being abridged, or reduced to least terms, by the common measure 7, the fractions become $\frac{1}{9}$, $\frac{1}{2}$. Numerator and denominator, multiplied by the abridged denominators, give respectively $\frac{2}{126}$ and $\frac{9}{126}$. Subtraction being made, the difference is $\frac{7}{126}$. This abridged by 7 is $\frac{1}{18}$.

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[1] *Bhága, ansa, vibhága, lava, &c.*, the numerator of a fraction. *Hara, hára, chheda, &c.*, the denominator of a fraction. That which is to be divided is the part (*ansa*); and that by which it is to be divided is *hara*, the divisor.—Gan. and Súr.

[2] *Rási*, a quantity, § 36.

[3] Among astronomers and other arithmeticians, oral instruction has taught to place the numerator above and the denominator beneath.—Gan.

No line is interposed in the original; but it has been introduced in the translation to conform to the modern practice. Bháskara subsequently directs (§ 36) an integer to be written as a fraction by placing under it unity for its denominator. The same is done by him in this place in the text.

31. Rule for the reduction of subdivided fractions[1]: half a stanza.

The numerators being multiplied by the numerators, and the denominators by the denominators, the result is a reduction to homogeneous form in subdivision of fractions.

[This is the ordinary rule for reducing a compound fraction to a simple fraction.]

32. Example. The quarter of a sixteenth of the fifth of three-quarters of two-thirds of a moiety of a *dramma* was given to a beggar by a person, from whom he asked alms: tell me how many cowry shells the miser gave, if thou be conversant, in arithmetic, with the reduction termed subdivision of fractions.

Statement: $\frac{1}{1}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{1}{5}$ $\frac{1}{16}$ $\frac{1}{4}$.

Reduced to homogeneousness, $\frac{6}{7680}$, or in least terms. $\frac{1}{1280}$.

Answer: A single cowry shell was given.[2]

33. Rule for the reduction of quantities increased or decreased by a fraction:[3] a stanza and a half.

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[1] *Prabhága-játi*, assimilation of sub-fractions, or making uniform the fraction of a fraction.—Gan.

Prabhága, a divided fraction or fraction of a fraction: as a part of a moiety, and so forth.—Gang.

[2] For a cowry shell is in the tale of money the 1280th part of a *dramma*, § 2.

[3] *Bhágánubandhajáti*, assimilation of fractional increase, reduction to uniformity of an increase by a fraction, or the addition of a part; from *anubandha*, junction.—Gan. *Bhágápaváhajáti*, assimilation of fractional decrease; from *apaváha*, deduction.—Gan.

These, as remarked by Ganesa, are merely particular cases of addition and subtraction. The fractions may be parts of an integer, or parts of the proposed quantity itself. Hence we get two sorts of each, named by Gangádhara and Súryadása, *rúpa-bhágánubandha*, addition of the fraction of a unit; *rúpa-bhágápaváha*, subtraction of the fraction of a unit; *rási-bhágánubandha*, addition of a fraction of the quantity; *rási-bhágápaváha*, subtraction of a fraction of the quantity.

The integer being multiplied by the denominator, the numerator is made positive or negative,[1] provided parts of a unit be added or be subtractive. But if indeed the quantity be increased or diminished by a part of itself, then, in the addition and subtraction of fractions, multiply the denominator by the denominator standing underneath[2], and the numerator by the same augmented or lessened by its own numerator.

[The first part of the rule follows from the identity, $a \pm \frac{b}{c} = \frac{ac \pm b}{c}$, and the second part, from the identities, $\frac{a}{b} \pm \frac{c}{d} \times \frac{a}{b} = \frac{ad \pm ac}{bd} = \frac{a(d \pm c)}{bd}$.

Hence, if we write $\frac{c}{d}$ underneath $\frac{a}{b}$, we get the second part of the rule.

The process may be repeated, if necessary.

Thus, $\frac{a}{b} + \frac{c}{d} \times \frac{a}{b} + \frac{e}{f}\left(\frac{a}{b} + \frac{c}{d} \times \frac{a}{b}\right) = \frac{a(d+c)(f+e)}{bdf}$.

An application of this last formula occurs in the examples given in § 35.]

34. Example. Say how much two and a quarter, and three less a quarter, are, when reduced to uniformity, if thou be acquainted with fractional increase or decrease.

Statement : 2 3
$\frac{1}{4}$ $\cdot\frac{1}{4}$

Reduced to homogeneousness, they become $\frac{9}{4}$ and $\frac{11}{4}$.

[In the original a dot ($\cdot$) is used instead of the sign *minus* ($-$).]

---

[1] *Dhana*, positive ; *rina*, negative.

[2] Indian arithmeticians write fractions under the quantities to which they are additive, or from which they are subtractive. Accordingly, the numerators and denominators are put in their order, one under the other.

35. Example. How much is a quarter added to its third part, with a half of the sum, and how much are two-thirds lessened by one-eighth of them, and then diminished by three-sevenths of the residue? Tell me likewise, how much half less its eighth part, added to nine-sevenths of the residue is, if thou be skilled, dear woman, in fractional increase and decrease.

Statement: $\frac{1}{4}$, $\frac{2}{3}$, $\frac{1}{2}$
$\frac{1}{3}$, $\cdot\frac{1}{8}$, $\cdot\frac{1}{8}$
$\frac{1}{2}$, $\cdot\frac{3}{7}$, $\frac{9}{7}$

Reduced to uniformity, the results are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{1}$.

[In the above examples we may apply the last formula given in § 33. Thus in the first example, we have $a=1$, $b=4$, $c=1$, $d=3$, $e=1$, $f=2$, and all the signs are *plus*. Hence the result is $\frac{1\times(3+1)\times(2+1)}{4\times3\times2}=\frac{12}{24}=\frac{1}{2}$; and similarly for the other two. The same process is exhibited in the *Manoranjana*.]

The Eight Rules of Arithmetic applied to Fractions.[1]

36. Rule for addition and subtraction of fractions[2]: half a stanza. The sum or (in the case of subtraction) the difference of fractions having a common denominator, is (taken). Unity[3] is put denominator of a quantity[4] which has no divisor.[5]

[This rule and the others which follow are all ordinary rules.]

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[1] *Bhinna-parikarmáshtaka*, the eight modes of process, as applicable to fractions: the preceding Section relating to those arithmetical processes as applicable to integers (*abhinna-parikarmáshtaka*.)

[2] *Bhinna-sankalita*, addition of fractions; *bhinna-vyavakalita*, subtraction of fractions.

[3] *Rúpa*, the species or form; anything having bounds.—Gang. In the singular, the arithmetical unit; in the plural, any integer.

[4] *Rási*, a congeries; a heap of things, of which unity is the scale of numeration; a quantity or number.

[5] That is, it is put denominator of an integer.

37. Example. Tell me, dear woman, quickly, how much a fifth, a quarter, a third, a half, and a sixth, make when added together. Say instantly what the residue of three is, subtracting those fractions.

Statement: $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{6}$.

Added together the sum is $\frac{29}{20}$.

Subtracting those fractions from three, the remainder is $\frac{31}{20}$.

38. Rule for multiplication of fractions[1]: half a stanza.

The product of the numerators, divided by the product of the denominators, (gives a quotient, which) is the result of multiplication of fractions.

39. Example. What is the product of two and a seventh, multiplied by two and a third, and of a moiety multiplied by a third? Tell, if thou be skilled in the method of multiplication of fractions.

Statement: 2 2
$\frac{1}{3}$ $\frac{1}{7}$.

Reducing to uniformity we get $\frac{7}{3}$, $\frac{15}{7}$. The product is $\frac{5}{1}$.

Statement: $\frac{1}{2}$ $\frac{1}{3}$.

The product is $\frac{1}{6}$.

40. Rule for division of fractions[2]: half a stanza.

After reversing the numerator and denominator of the divisor, the remaining process for division of fractions is that of multiplication.

41. Example. Tell me the result of dividing five by two and a third; and a sixth by a third; if thy

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[1] *Bhinna-guṇana*, multiplication of fractions.

[2] *Bhinna-bhágahára*, division of fractions.

understanding, sharpened into confidence, be competent for the division of fractions.[1]

Statement : $\begin{matrix} 2 \\ \frac{1}{3} \end{matrix}$ $\left(\frac{7}{3}\right)$ $\frac{5}{1}$. $\frac{1}{3}$, $\frac{1}{6}$.

Proceeding as directed, the quotients are $\frac{15}{7}$ and $\frac{1}{2}$.

42. Rule for involution and evolution of fractions[2]: half a stanza.

If the square be sought, find both squares; if the cube be required, both cubes : or, to discover the root (of cube or square), extract the roots of both (numerator and denominator).

43. Example. Tell me quickly the square of three and a half ; and the square root of the square; and the cube of the same ; and the cube root of that cube ; if thou be conversant with fractional squares and roots.

Statement : $\begin{matrix} 3 \\ \frac{1}{2} \end{matrix}$ or reduced $\frac{7}{2}$.

Its square is $\frac{49}{4}$; of which the square root is $\frac{7}{2}$. The cube of it is $\frac{343}{8}$; of which again the cube root is $\frac{7}{2}$.

Section IV.

Cipher.[3]

44—45. Rule for arithmetical process relative to cipher : two couplets.

In addition, cipher makes the sum equal to the additive.[4] In involution and (evolution) the result is cipher. A definite quantity,[5] divided by cipher, is the

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[1] Ganesa omits the latter half of the stanza. Gangádhara gives it entire

[2] *Bhinna-varga*, square of a fraction ; *bhinna-ghana*, cube of a fraction.

[3] *Súnya*, *kha*, and other synonyms of vacuum or etherial space ; nought or cipher ; a blank or the privation of specific quantity.—Krishna on *Víja-ganita*.

[4] *Kshepa*, that which is cast or thrown in ; additive.—Gang.

[5] Rási. See § 36.

submultiple of nought.[1] The product of cipher is nought: but it must be retained as a multiple of cipher,[2] if any further operation impend. Cipher having become a multiplier, should nought afterwards become a divisor, the definite quantity must be understood to be unchanged. So likewise any quantity, to which cipher is added, or from which it is subtracted, (is unaltered).

[The first four rules are clear. The rule, *viz.*, "cipher having become a multiplier, &c.," is not accurate. For $\frac{a \times 0}{0} = \frac{0}{0}$ = indeterminate, and $not = a$, as the rule says. The idea of *infinity* is not introduced here by the author. It is, however, introduced by him in the *Víja-ganita*, and also by Ganesa in his commentary on the above couplets.]

46. Example. Tell me how much cipher added to five is, and the square of cipher, and its square root, its cube, and cube root; and five multiplied by cipher; and how much ten is subtracting cipher; and what number it is, which multiplied by cipher, and added to half itself, and multiplied by three, and divided by cipher, amounts to a given number sixty-three.

Statement: 0. Cipher added to 5 makes 5. Square of cipher, 0. Square root, 0. Cube of cipher, 0. Cube root, 0.

---

[1] *Kha-hara*, a fraction with cipher for its denominator. According to the remark of Ganesa, it is an infinite quantity: since it cannot be determined how great it is. It remains unaltered by the addition or subtraction of finite quantities: since, in the preliminary operation of reducing both fractional expressions to a common denominator, preparatory to taking their sum or difference, both numerator and denominator of the finite quantity vanish. Ranganátha affirms that it is infinite, because the smaller the divisor is, the greater is the quotient: now cipher, being in the utmost degree small, gives a quotient infinitely great.

[2] *Khaguna*, a quantity which has cipher for its multiplier. Cipher is set down by the side of the multiplicand, to denote it.—Gan.

Statement : 5. This multiplied by cipher makes 0.

Statement : 10. This divided by cipher gives $\frac{10}{0}$.

Statement : an unknown quantity; its multiplier, 0; additive, $\frac{1}{2}$; multiplicator, 3 ; divisor, 0 ; given number, 63 ; assumption, 1.

Then, either by inversion or position, as subsequently explained (§47 and §50), the number is found, 14. This mode of computation is of frequent use in astronomical calculation.

[The last example as translated by Colebrooke appears to be meaningless and absurd. If we put x for the required number, we get the equation,

$$\frac{3\ (x \times 0 + \frac{1}{2}x)}{0} = 63,$$

which is manifestly absurd. The correct translation, however, would lead to the equation,

$$\frac{0 \times (x + \frac{1}{2}x) \times 3}{0} = 63,$$

of which $x = 14$ is a solution.]

CHAPTER III.
MISCELLANEOUS RULES.

SECTION I.
INVERSION.

47—48. Rule of inversion[2]: two stanzas. To investigate a quantity, one being given,[3] make the divisor a multiplicator; and the multiplier a divisor; the square, a root; and the root, a square[4]; turn the negative into positive, and the positive into negative. If a quantity is to be increased or diminished by its own proportionate part, let the (lower[5]) denominator, being increased or diminished by its numerator, become the (corrected[6]) denominator, and the numerator remain unchanged; and then proceed with the other operations of inversion, as before directed.

[The reason for the rule is clear from the example given in § 49. If we want an arithmetical solution of such a problem, we must begin from the end, and invert every operation indicated in the problem. If a quantity is to be increased or diminished by its own proportionate part, *i.e.*, if we have an equation

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[1] *Prakirna*, miscellaneous. The rules contained in the first five sections of this chapter have none answering to them in the Arithmetic of Brahmagupta and Sridhara.

[2] *Viloma-vidhi*, *Viloma-kriyá*, *Vyasta-vidhi*, inversion.

[3] *Drisya*, the quantity or number, which is visible; the given quantity.

[4] And the cube, a cube root; and the cube root, a cube.—Gan.

[5] Gangádhara.

[6] Gangádhara.

of the form $x\left(1\pm\frac{a}{b}\right)=c$, then evidently $x=\frac{bc}{b\pm a}=c\mp\frac{ac}{b\pm a}$. This explains the latter part of the rule.]

49. Example. Pretty girl, with tremulous eyes, if thou know the correct method of inversion, tell me the number, which multiplied by three, and added to three-quarters of the product, and divided by seven, and reduced by subtraction of a third part of the quotient, and then multiplied into itself, and having fifty-two subtracted from the product, and the square root of the remainder extracted, and eight added, and the sum divided by ten, yields two.[1]

Statement: Multiplier 3. Additive $\frac{3}{4}$. Divisor 7. Decrease $\frac{1}{3}$. Square —. Subtractive 52. Square root —. Additive 8. Divisor 10. Given number 2.

Proceeding as directed, the result is 28, the number sought.

Section II.

Supposition.

50. Rule of supposition[2]: one stanza. Any number assumed at pleasure is treated as specified in the particular question, being multiplied and divided, raised or diminished by fractions; then the given quantity, being multiplied by the assumed number and

[1] All the operations are inverted. The known number 2, multiplied by the divisor 10 converted into a multiplicator, makes 20; from which the additive 8, being subtracted, leaves 12; the square whereof (extraction of the root being directed) is 144; and adding the subtractive 52, it becomes 196; the root of this (square being directed) is 14; added to its half, 7, it amounts to 21, which multiplied by 7, is 147. This again divided by 7 and multiplied by 3 makes 63, which, subtracted from 147, leaves 84; and this divided by 3, gives 28.—*Mano.*

[2] *Ishta-karman*, operation with an assumed number. It is the rule of false position, supposition, and trial and error.

divided by that (which has been found), yields the number sought. This is called the process of supposition.

51. Example. What is that number, which multiplied by five, and having the third part of the product subtracted, and the remainder divided by ten, and one-third, a half and a quarter of the original quantity added, gives two less than seventy ?

Statement: Multiplier 5. Subtractive $\frac{1}{3}$ of itself. Divisor 10. Additive $\frac{1}{3}$ $\frac{1}{2}$ $\frac{1}{4}$ of the quantity. Given 68.

Putting 3; this multiplied by 5 is 15; less its third part, is 10; divided by 10, yields 1. Added to the third, half and quarter of the assumed number 3, *viz.*, $\frac{3}{3}$ $\frac{3}{2}$ $\frac{3}{4}$, the sum is $\frac{17}{4}$. By this divide the given number 68 taken into the assumed one 3; the quotient is 48.

The answer is the same with any other assumed number, as 1, &c.

Thus, by whatever number the quantity is multiplied or divided in any example, or by whatever fraction of the quantity it is increased or diminished, by the same should the like operations be performed on a number arbitrarily assumed; and by that, which results, divide the given number taken into the assumed one; the quotient is the quantity sought.

[The rule in § 50 is a clumsy way of solving a simple equation. The reason for it will appear below.

Let x denote the number sought in § 51.

Then, $5x\times\frac{2}{3}\times\frac{1}{10}+(\frac{1}{3}+\frac{1}{2}+\frac{1}{4})x=68.$

Multiply both sides by any assumed integer k, and we get

$$x=\{68\times k\}\div\{k\times 5\times\tfrac{2}{3}\times\tfrac{1}{10}+(\tfrac{1}{3}+\tfrac{1}{2}+\tfrac{1}{4})k\}.$$

Thus we see that there is no need at all of assuming an integer k, as the rule directs.]

52. Example of reduction of a given quantity.[1] Out of a heap of pure lotus flowers, a third part, a fifth and a sixth were offered respectively to the gods Siva, Vishnu and the Sun; and a quarter was presented to Bhavâní. The remaining six lotuses were given to the venerable preceptor. Tell quickly the whole number of lotuses.

Statement: $\frac{1}{3}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{4}$; known 6. Putting one for the assumed number, and proceeding as above, the quantity is found 120.

[In Pandit Jívánanda Vidyáságara's edition, there is an example before this, which is omitted by Colebrooke. It is as follows:—

Out of a herd of elephants, half together with a third part of itself was roaming in a forest; a sixth part together with a seventh of itself was drinking water in a river; and an eighth part together with a ninth of itself was playing with lotuses. The leader of the herd was seen accompanied by three females. What was the number of elephants in the herd?

This may be solved either by an arithmetical or an algebraical method, both being practically the same. Adopting the latter, and putting x for the required number, we get the equation $\{\frac{1}{2}(1+\frac{1}{3})+\frac{1}{6}(1+\frac{1}{7})+\frac{1}{8}(1+\frac{1}{9})\}x+4=x$, whence $x=1008$.

It may be observed here that all the examples in this Section are problems producing simple equations, which are solved not by the ordinary method of solving simple equations, but by the author's method stated in § 50. They may also be worked out by a purely arithmetical method.

The algebraical solution of the problem in § 52 is as follows:—

Let x denote the whole number of lotuses.

Then $(\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{4})x+6=x$, whence $x=120$].

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[1] *Drisya-játi*, reduction of the visible or given quantity with fractions affirmative or negative; here, with negative; in the preceding example, with affirmative.

53. Example of reduction of residues.[1] A traveller, engaged in a pilgrimage, gave half his money at *Prayága;* two-ninths of the remainder at *Kásí;* a quarter of the residue in payment of taxes on the road; six-tenths of what was left at *Gayá;* there remained sixty-three *nishkas*, with which he returned home. Tell me the amount of his original stock of money, if you have learned the method of reduction of fractions of residues.

Statement: $\frac{1}{2}$ $\frac{2}{9}$ $\frac{1}{4}$ $\frac{6}{10}$; known 63. Putting one for the assumed number, subtracting the numerator from its denominator, multiplying denominators together, and in other respects proceeding as directed, the remainder is found $\frac{7}{60}$. By this dividing the given number 63 taken into the assumed quantity, the original sum comes out 540.

Or it may be found by the method of reduction of fractional decrease (§ 33).

Statement: $\frac{1}{1}.\frac{1}{2}.\frac{2}{9}.\frac{1}{4}.\frac{6}{10}$. Being reduced to homogeneous form, the result is $\frac{7}{60}$: whence the sum is deduced 540.

Or this may also be found by the rule of inversion (§ 47).

[Let x denote the original stock in *nishkas*. Then, at *Prayága*, there remained $\frac{1}{2}x$; at *Kásí*, $\frac{2}{9}$ of this being spent, there remained $\frac{7}{9}$ of $\frac{1}{2}x$; similarly, on the road, there remained $\frac{3}{4}$ of $\frac{7}{9}$ of $\frac{1}{2}x$, and at *Gayá*, $\frac{4}{10}$ of $\frac{3}{4}$ of $\frac{7}{9}$ of $\frac{1}{2}x$ or $\frac{7}{60}x$; hence we get $\frac{7}{60}x = 63$, whence $x = 540$.]

54. Example of reduction of differences.[2] Out of a swarm of bees, one-fifth part settled on a blossom of

[1] *Sesha-játi*, assimilation of residue; reduction of fractions of residues or successive fractional remainders.

[2] *Vislesha-játi*, assimilation of difference; reduction of fractional differences.

Kadamba,[1] and one-third on a flower of *silíndhrí*[2]; three times the difference of those numbers flew to the bloom of a *Kutaja*.[3] One bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a jasmine and pandanus. Tell me, charming woman, the number of bees.

Statement : $\frac{1}{5}$ $\frac{1}{3}$ $\frac{2}{15}$; known quantity, 1 ; assumed, 30.

A fifth of the assumed number is 6; a third is 10; difference 4 ; multiplied by 3 gives 12 ; and the remainder is 2. Then the product of the known quantity by the assumed one, being divided by this remainder shows the number of bees 15.

Here also putting unit for the assumed quantity, the number of the swarm is found 15.

So in other instances likewise.[4]

[Let x denote the number of bees.

Then, $\frac{1}{5}x+\frac{1}{3}x+3(\frac{1}{3}x-\frac{1}{5}x)+1=x$, whence $x=15$.]

Section III.

55. Rule of concurrence : half a stanza.

The sum with the difference added and subtracted,

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[1] *Kadamba*, Nauclea orientalis or N. Kadamba.

[2] *Silíndhrí*, a plant resembling the *Kachora*.—Krishna on *Víja-ganita*.

[3] Echites antidysenterica.

[4] The *Manoranjana* introduces one more example, which is there placed after the second, and is here subjoined :—" The third part of a necklace of pearls, broken in an amorous struggle, fell to the ground ; its fifth part rested on the couch ; the sixth part was saved by the wench ; and the tenth part was taken up by her lover ; six pearls remained strung. Say of how many pearls the necklace was composed."

Statement : $\frac{1}{3}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{10}$. Rem. 6.

Answer : 30.

being halved, gives the two quantities. This is termed concurrence.[1]

[Let x and y denote the required numbers. Then,
$x+y=k$, $x-y=l$, where k and l are given quantities ;
whence $x=\frac{1}{2}(k+l)$, $y=\frac{1}{2}(k-l)$.]

56. Example. Tell me the numbers, the sum of which is a hundred and one, and the difference, twenty-five ; if thou know the rule of concurrence, dear child.

Statement : sum 101 ; difference 25.

The two numbers are 38 and 63.

57. Rule of dissimilar operation[2] ; half a stanza.

The difference of the squares, divided by the difference of the radical quantities, gives their sum[3] ; whence the quantities are found in the mode before directed.

[Let x and y be the numbers. Then,

$$x^2-y^2=m,\ x-y=n.$$

$$\therefore x+y=\frac{m}{n}, \text{ \&c.}]$$

58. Example. Tell me quickly, skilful calculator, what numbers they are, of which the difference is eight, and the difference of squares four hundred.

Statement : difference of the quantities 8 ; difference of the squares 400.

The numbers are 21 and 29.

---

[1] *Sankramana*, concurrence or mutual penetration in the shape of sum and difference.—Gang. Investigation of two quantities concurrent or grown together in the form of sum and difference.—Gan. Calculation of quantities latent within those exhibited.—Súr.

[2] *Vishama-karman*, the finding of the quantities, when the difference of their squares is given, and either the sum or the difference of the quantities.—Gan. A species of concurrence.—Gang. See below. §135.

[3] Or divided by their sum, gives their difference.—Gan.

Section IV.

Problem concerning Squares.[1]

A certain problem relative to squares is propounded in the next instance.

59. Rule. The square of an arbitrary number, multiplied by eight and lessened by one, then halved and divided by the assumed number, is one quantity; its square, halved and added to one, is the other. Or unity, divided by double an assumed number and added to that number, is a first quantity; and unity is the other. These give pairs of quantities, the sum and difference of whose squares, lessened by one, are squares.

[Let n be the assumed number. Then, by the first part of the rule, the two numbers are,

$$\frac{1}{2n}(8n^2-1) \text{ and } \tfrac{1}{2}\left\{\frac{1}{2n}(8n^2-1)\right\}^2+1.$$

The sum of the squares of these numbers lessened by 1 is

$$\tfrac{1}{4}\left(4n-\frac{1}{2n}\right)^4+2\left(4n-\frac{1}{2n}\right)^2$$

$$=\left(4n-\frac{1}{2n}\right)^2\left\{\tfrac{1}{4}\left(4n-\frac{1}{2n}\right)^2+2\right\}$$

$$=\left(4n-\frac{1}{2n}\right)^2\left(2n+\frac{1}{4n}\right)^2,$$ a perfect square. Similarly, the difference of the squares of the numbers lessened by 1 is a perfect square.

Again, by the second part of the rule, the numbers are $\frac{1}{2n}+n$ and 1; and $\left\{\left(\frac{1}{2n}+n\right)^2\pm(1)^2\right\}-1=\left(\frac{1}{2n}\pm n\right)^2$, which are perfect squares.

Hence the reason for the rule is evident.]

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[1] *Varga-karman*, operation relative to squares; an indeterminate problem, admitting innumerable solutions.

60. Tell me, my friend, numbers, the sum and difference of whose squares, less one, afford square roots, which dull smatterers in algebra labour to excruciate, puzzling for it in the six-fold method of discovery there taught.[1]

To bring out an answer by the first rule, let the number put be $\frac{1}{2}$. Its square, $\frac{1}{4}$, multiplied by 8, is 2; which lessened by 1 is 1. This halved is $\frac{1}{2}$, and divided by the assumed number $\frac{1}{2}$ gives 1 for the first quantity. Its square halved is $\frac{1}{2}$, which, added to 1, makes $\frac{3}{2}$. Thus the two quantities are 1 and $\frac{3}{2}$.

So, putting 1 for the assumed number, the numbers obtained are $\frac{7}{2}$ and $\frac{57}{8}$. With the supposition of 2, they are $\frac{31}{4}$ and $\frac{993}{32}$.

By the second method, let the assumed number be 1. Unity divided by the double of it is $\frac{1}{2}$, which added to the assumed number makes $\frac{3}{2}$. The first quantity is thus found. The second is unity. With the supposition of 2, the quantities are $\frac{9}{4}$ and 1. Putting 3, they are $\frac{19}{6}$ and 1.

61. Another Rule.[2] The square of the square of an arbitrary number, and the cube of that number, respectively multiplied by eight, adding one to the first product, are such quantities, equally in arithmetic and in algebra.

Put $\frac{1}{2}$. The square of the square of the assumed number is $\frac{1}{16}$, which multiplied by 8 makes $\frac{1}{2}$. This

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[1] This question, found in some copies of the text, and interpreted by Gangádhara and the *Manoranjana*, is unnoticed by the other commentators. [We do not know what the author means by *the six-fold method of discovery*. Colebrooke does not say anything about it.—Ed.]

[2] To bring out answers in whole numbers, the two preceding solutions giving fractions.—Gan. and Súr.

added to 1 is $\frac{3}{2}$, which is the first quantity. Again put $\frac{1}{2}$. Its cube is $\frac{1}{8}$, which multiplied by 8 gives the second quantity 1. Next supposing 1, the two quantities are 9 and 8. Assuming 2, they are 129 and 64. Putting 3, they are 649 and 216. And so on, without end, by means of various suppositions, in the several foregoing methods.

It is said that algebraic solution similar to arithmetical rules appears obscure; but it is not so to the intelligent; nor is it six-fold, but manifold.

[Let n be the arbitrary number. Then, by the rule, the numbers are, $8n^4 + 1$ and $8n^3$;
and $\{(8n^4+1)^2 \pm (8n^3)^2\} - 1 = (4n^2)^2 \ (2n^2 \pm 1)^2$, which are perfect squares.

Hence the reason for the rule is obvious.]

Section v.

62—63. Rule for assimilation of the root's coefficient[1]: two stanzas.

The sum or difference of a quantity and of a multiple of its square root being given, the square of half the coefficient[2] is added to the given number, and the square root of their sum (is extracted; that root,) with half the coefficient added or subtracted, being squared, is the quantity sought by the interrogator. If the quantity have a fraction (of itself) added or subtracted, divide the number given and the multiplicator of the root, by unity increased or lessened by the fraction,

[1] *Múla-játi. múla-gunaka-játi* or *ishta-múlánsa-játi*, assimilation and reduction of the root's coefficient with a fraction.

[2] *Guna*, multiplicator; *múla-guna*, root's multiplier, the coefficient of the root.

and the required quantity may be then discovered, proceeding with those quotients as above directed.

A quantity, increased or diminished by its square root multiplied by some number, being given, add the square of half the multiplier of the root to the given number; and extract the square root of the sum. Add half the multiplier, if the difference were given; or subtract it, if the sum were so. The square of the result will be the quantity sought.

[The third paragraph is in prose in the original, and is added by the author by way of explanation of the two preceding metrical rules.

Suppose we have the equations,

$$x \pm a\sqrt{x} = b \text{ . . . . } (1).$$

Then, completing the square, we get

$$x \pm a\sqrt{x} + \left(\frac{a}{2}\right)^2 = b + \left(\frac{a}{2}\right)^2;$$

$$\therefore \sqrt{x} = \sqrt{b + \left(\frac{a}{2}\right)^2} \mp \frac{a}{2};$$

$$\therefore \quad x = \left\{\sqrt{b + \left(\frac{a}{2}\right)^2} \mp \frac{a}{2}\right\}^2.$$

Hence the reason for the first part of the rule is clear. It is the ordinary rule for solving an equation reducible to a quadratic by completing the square.

The second part of the rule is meant for equations of the form

$$x \pm \frac{c}{d}x \pm a\sqrt{x} = b \text{ . . . . . . } (2),$$

whence we get

$$x \pm \frac{a\sqrt{x}}{1 \pm \frac{c}{d}} = \frac{b}{1 \pm \frac{c}{d}},$$

which is of the form (1), and may be solved as above. Thus we see the reason for the second part of the rule.]

64. Example: the root subtracted, and the difference given. One pair out of a flock of geese remained

sporting in the water, and saw seven times the half of the square root of the flock proceeding to the shore tired of the diversion. Tell me, dear girl, the number of the flock.

Statement: coeff. $\frac{7}{2}$; given 2. Half the coefficient is $\frac{7}{4}$; its square $\frac{49}{16}$ added to the given number, makes $\frac{81}{16}$, the square root of which is $\frac{9}{4}$. Half the coefficient being added, the sum is $\frac{16}{4}$; or, reduced to least terms, 4. This squared is 16; the number of the flock; as required.

[Let x denote the number of the flock.

Then, $2+\frac{7}{2}\sqrt{x}=x$;

$$\therefore\ x-\frac{7}{2}\sqrt{x}+\left(\frac{7}{4}\right)^2=2+\left(\frac{7}{4}\right)^2=\left(\frac{9}{4}\right)^2;$$

$$\therefore\ \sqrt{x}-\frac{7}{4}=\frac{9}{4};$$

$$\text{whence } x=\left(\frac{7}{4}+\frac{9}{4}\right)^2=16.$$

This is an instance of the first part of the preceding rule.]

65. Example: the root added, and the sum given. Tell me what the number is, which, added to nine times its square root, amounts to twelve hundred and forty.

Statement: coeff. 9; given 1240. Proceeding by the rule, the required number is 961.

[Let x denote the number required. Then, $x+9\sqrt{x}=1240$, whence x.]

66. Example: the root and a fraction both subtracted. Of a flock of geese, ten times the square root of the number departed for the *Mánasa* lake,[1] on the

Wild geese are observed to quit the plains of India, at the approach of the rainy season; and the lake called *Mánasasarovara* is covered with water-fowl, especially geese, during that season. The *Hindus* suppose the whole tribe of geese to retire to the holy lake at the approach of rain. The bird is sacred to Brahmá. [See *Raghuvansa*, XIII. 55.—Ed.]

approach of a cloud : an eighth part went to a forest of *Sthalapadminís*[1] : three couples were seen engaged in sport on the water abounding with delicate fibres of the lotus. Tell, dear girl, the whole number of the flock.

Statement : coeff. 10 ; fraction $\frac{1}{8}$; given 6.

Proceeding by the (second) rule (§63), unity, less the fraction, is $\frac{7}{8}$; and the coefficient and the given number, being both divided by that, become $\frac{80}{7}$ and $\frac{48}{7}$; and the half coefficient is $\frac{40}{7}$. With these, proceeding by the (first) rule (§62), the number of the flock is found 144.

[Let x denote the whole number of the flock.

$$\text{Then, } 10\sqrt{x}+\frac{1}{8}x+6=x;$$

$$\therefore \frac{7}{8}x-10\sqrt{x}=6\text{ ; whence } x.$$

This is an instance of the second part of the rule in §62—63.]

67. Example. The son of Prithá,[2] irritated in fight, shot a quiver of arrows to slay Karna. With half his arrows, he parried those of his antagonist ; with four times the square root of the quiver-full, he killed his horses ; with six arrows, he slew Salya[3] ; with three he demolished the umbrella, standard and bow ; and with one, he cut off the head of the foe. How many were the arrows which Arjuna let fly ?

Statement : fraction $\frac{1}{2}$; coeff. 4 ; given 10.

The given number and coefficient being divided by unity less the fraction become 20 and 8 ; and proceeding by the rule (§62), the number of arrows comes out 100.

---

[1] The plant intended is not ascertained. The context would seem to imply that it is arboreous, as the term signifies forest.

[2] Arjuna, surnamed Pártha ; his matronymic from Prithá or Kuntí.

[3] One of the *Kauravas*, and charioteer of Karna.

[Let x denote the number of arrows.

Then, $\frac{1}{2}x + 4\sqrt{x} + 6 + 3 + 1 = x$, whence x.]

68. Example. The square root of half the number of a swarm of bees is gone to a shrub of jasmin[1]; and so are eight-ninths of the whole swarm : a female is buzzing to one remaining male that is humming within a lotus in which he is confined, having been allured to it by its fragrance at night.[2] Say, lovely woman, the number of bees.

Here eight-ninths of the quantity and the root of its half are negative (and consequently subtractive) from the quantity : and the given number is two of the specific things. The negative quantity, and the given number halved, bring out half the quantity sought.[3] Thus :—

Statement : fraction $\dot{\frac{8}{9}}$; coeff. $\dot{\frac{1}{2}}$; given 1.

A fraction of half the quantity is the same as half the fraction of the quantity ; the fraction is therefore set down (unaltered).

Here proceeding as above directed, there comes out half the quantity, 36; which being doubled is the number of bees in the swarm, 72.

---

[1] *Málati*, jasminum grandiflorum.

[2] The lotus being open at night and closed in the day, the bee might be caught in it.—Gan.

[3] In such questions it is necessary to observe whether the coefficient of the root be so of the root of the whole number, or of that of its part ; for that quantity is found, of whose root the coefficient is used. But in the present case, the root of half the quantity is proposed ; and accordingly, the half of the quantity will be found by the rule. The number given, however, belongs to the entire quantity. Therefore, taking half the given number, half the required number is to be brought out by the process before directed.—*Mano.* and Súr.

[Let x denote the number of bees.

Then, $\sqrt{\frac{1}{2}x} + \frac{8}{9}x + 2 = x$.

Put $y = \frac{1}{2}x$, and we get

$$y - \frac{8}{9}y - \frac{1}{2}\sqrt{y} = 1,$$

whence by the rule in §§ 62-63, we obtain $y = 36$, and $\therefore$ $x = 72$.

Thus the reason for the process given in the text is clear. The reason given by the author and the commentators is not very clear.]

69. Example: the root and a fraction both added. Find quickly, if thou have skill in arithmetic, the quantity which added to its third part and eighteen times its square root, amounts to twelve hundred.

Statement: fraction $\frac{1}{3}$; coeff. 18; given 1200.

Here, dividing the coefficient and given number by unity added to the fraction (§63), and proceeding as before directed, the number is brought out, 576.

[Let x denote the number required.

Then, $x + \frac{1}{3}x + 18\sqrt{x} = 1200$; whence x.]

Section VI.

Rule of Proportion.[1]

70. Rule of three terms[2]: one stanza.

The first and last terms, which are the argument and requisition, must be of like denomination; the fruit, which is of a different species, stands between them: and that, being multiplied by the demand and divided

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[1] [A more literal translation would be 'Rule of Three,' the word in the original being *trairáśika*.—Ed.]

[2] *Trairáśika*, calculation belonging to a set of three terms.—Gang. Rule of Three. The first term is *pramána*, the measure or argument; the second is its fruit, *phala*, or produce of the argument; the third is *ichchhá*, the demand, requisition, desire or question.—Gan.

by the first term, gives the fruit of the demand.[1] In the inverse method, the operation is reversed.[2]

[The rule is the ordinary mechanical one for solving problems involving the Rule of Three, direct and inverse. It is not stated in the light of the principle of proportion, and is practically the same as that given in Mr. Barnard Smith's well-known work on Arithmetic, Art. 155.]

71. Example. If two and a half *palas* of saffron be obtained for three-sevenths of a *nishka*, say instantly, best of merchants, how much is got for nine *nishkas*.

Statement : $\frac{3}{7}$ $\frac{5}{2}$ $\frac{9}{1}$. Answer : 52 *palas* and 2 *karshas*.

[This is an example of the Rule of Three direct. Worked out by the principle of proportion, the process will stand thus :—

Let x denote the quantity sought in *palas*.

Then, we evidently have the proportion,

$$\tfrac{5}{2} : x :: \tfrac{3}{7} : 9,$$

$$\therefore x \times \tfrac{3}{7} = \tfrac{5}{2} \times 9,$$

$$\therefore \quad x = \frac{\frac{5}{2} \times 9}{\frac{3}{7}} = 52\tfrac{1}{2}.$$

Thus the answer is $52\frac{1}{2}$ *palas* = 52 *palas*, 2 *karshas*.

The reason for the rule in § 70 is obvious.]

72. Example. If one hundred and four *nishkas* are got for sixty-three *palas* of best camphor, consider and tell me, friend, what may be obtained for twelve and a quarter *palas*.

Statement: 63 104 $\frac{49}{4}$. Answer: 20 *nishkas*, 3 *drammas*, 8 *panas*, 3 *kákinís*, 11 cowryshells and $\frac{1}{9}$th part.

[This also is an instance of the Rule of Three direct, and may be worked out as above.]

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[1] *Ichchhá-phala*, produce of the requisition, or fruit of the question ; it is of the same denomination or species with the second term.

[2] See § 74.

73. Example. If a *khári* and one-eighth of rice may be procured for two *drammas*, say quickly what may be had for seventy *panas*.

Statement, reducing *drammas* to *panas*: 32 $\frac{9}{8}$ 70. Answer: 2 *kháris*, 7 *dronas*, 1 *ádhaka*, 2 *prasthas*.

[This is a third instance of the Rule of Three direct.]

74. Rule of Three inverse.[1]

If the fruit diminish as the requisition increases, or augment as that decreases, they who are skilled in accounts consider the Rule of Three terms to be inverted.[2]

When there is diminution of fruit, if there be increase of requisition, and increase of fruit if there be diminution of requisition, then the inverse Rule of Three is (employed.)

[This is the ordinary definition of inverse variation.]

75. For instance, when the value of living beings[3] is regulated by their age; and in the case of gold, where the weight and touch are compared[4]; or when heaps[5] are subdivided; let the inverted Rule of Three terms be (used).

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[1] *Vyasta-trairásika* or *Viloma-trairásika*, rule of three terms inverse.

[2] The method of performing the inverse rule has been already taught (§ 70), *viz.*, "in the inverse method, the operation is reversed;" *i.e.*, the fruit is to be multiplied by the argument and divided by the demand.—Súr.

When the fruit increases or decreases, as the demand is augmented or diminished, the direct rule (*Krama-trairásika*) is used; else the inverse.—Gan.

[3] Slaves and cattle. The price of the older is less; of the younger, greater.—Gang. and Súr.

[4] Colour on the touchstone. See Alligation, § 101.

[5] *See* Chap. X. When heaps of grain, which had been meted with a small measure, are again meted with a larger one, the number decreases; and when those, which had been meted with a large measure, are again meted with a smaller one, there is increase of number.—Gang. and Súr.

[Some instances of inverse variation are here mentioned. The reason is clear from the foot-notes appended. The author does not mention the common instance of time and agency required for a given piece of work.]

76. Example of age and price of living beings. If a female slave, sixteen years of age, bring thirty-two (*nishkas*), what will one aged twenty cost? If an ox which has been worked two years sell for four *nishkas*, what will one, which has been worked six years, cost?

1st Statement: 16 32 20. Answer: $25\frac{3}{5}$ *nishkas.*

2nd Statement: 2 4 6. Answer: $1\frac{1}{3}$ *nishka.*

[Let x denote the cost in the first example. Then $\because$ the greater is the age, the smaller is the cost, we have the proportion,

$$16 : 20 :: x : 32,$$

$$\text{Whence } x = \frac{16 \times 32}{20} = 25\tfrac{3}{5}.$$

Similarly the second example as well as those in the next two articles may be worked out.]

77. Example of touch and weight of gold. If a *gadyánaka* of gold of the touch of ten may be had for one *nishka* (of silver), what weight of gold of fifteen touch may be bought for the same price?

Statement: 10 1 15. Answer $\frac{2}{3}$

78. Example of subdivision of heaps. A heap of grain having been meted with a measure containing seven *ádhakas*, if a hundred such measures were found, what would be the result with one containing five *ádhakas*?

Statement: 7 100 5. Answer 140.

79. Rule of compound proportion[1]: one stanza.

[1] [A more literal translation would be, "Rule of Five and so forth," the word in the original being, *pancharásikádau.*—Ed.] This, which is the compound Rule of Three, comprises, according to Ganesa, two or more sets of three terms (*trairásika*); or two or more proportions (*anupáta*), as Súryadása

In the method of five, seven, nine or more[1] terms, transpose the fruit and divisors[2]; and the product of the larger set of terms, being divided by the product of the less set of terms[3], the quotient is the produce (sought)

[This is practically the same rule, rather incompletely stated, for solving problems involving the Double Rule of Three, as that given in Mr. Barnard Smith's work on Arithmetic, Art. 161.* It is a clumsy and a purely mechanical rule, having no connection whatever with the principle of proportion. The meaning of the phrase, "transpose the fruit and divisors," as explained by Ganesa, will appear from the foot-notes appended to the following articles. It should be observed here that "the product of the larger set" is not necessarily the numerically larger product; see the example in § 82. For

observes. Thus the rule of five (*pancha-rásika*) comprises two proportions; that of seven (*sapta-rásika*) three; that of nine (*nava-rásika*) four; and that of eleven (*ekádasa-rásika*) five.

[1] Meaning eleven. *Mano*. and Súr.

[2] Ganesa and the commentator of the *Vásaná* understand this last word (*chhid*, divisor) as relating to denominators of fractions; and the transposing of them (if any there be) is indeed right: accordingly the author gives under this rule an example of working with fractions (§ 81). But the *Manoranjana* and Súryadása explain it otherwise; and the latter cites as, ancient commentary entitled *Ganita-Kaumudí* in support of his exposition. "There are two sets of terms; those which belong to the argument, and those which appertain to the requisition. The fruit in the first set is called produce of the argument; that in the second is named divisor of the set. They are to be transposed, or reciprocally brought from one set to the other; *i.e.*, put the fruit in the second set, and the divisor in the first. Would it not be enough to say, transpose the fruits of both sets? The author of the *Kaumudí* replies, 'the designation of divisor serves to indicate that after transposition, the fruit of the second set being included in the product of the less set of terms, the product of the greater set is to be divided by it'. Some, however, intepret it as relative to fractions. But that is wrong: for the word would be superfluous." [This explanation is not very clear.—Ed.]

[3] *Bahu-rási-paksha*, set of many terms. That to which the fruit is brought is the larger set.—Gang. Or, if there be fruit on both sides, that, in which the fruit of the requisition is, is the larger set.—Gan. *Laghu-rási-paksha*, set of fewer terms.

the meaning of the term "larger set," see foot-note 3 p. 42. The phrase has been rather loosely used.]

80. Example. If the interest of a hundred for a month be five, say what the interest of sixteen is for a year. Find likewise the time from the principal and interest; and knowing the time and produce, tell the principal sum.

Statement:

1	12
100	16
5	

Answer[1]: the interest is $9\frac{3}{5}$.

To find the time; statement:

1	
100	16
5	$\frac{48}{5}$

Answer[2]: months 12.

To find the principal; statement:

1	12
100	
5	$\frac{48}{5}$

Answer[3]: principal 16.

[Worked out by the principle of proportion, the process will stand thus:—

Let x denote the interest required. Then, the int. of 16 for 1 year = int. of 16 × 12 for 1 month; and ∵ with a given time,

---

[1] Transposing the fruit,

1	12
100	16
	5

Product of the larger set, 960. Do. of the less set, 100. Quotient, $\frac{960}{100}$ or $\frac{48}{5}$.

[2] Transposing both fruits,

1	
100	16
$\frac{48}{5}$	5

and the denominator,

1	
100	16
48	5
	5

Product of the larger set, 4800. Do. of the less set, 400. Quotient, 12.

[3] Transposing both fruits,

1	12
100	
$\frac{48}{5}$	5

and the denominator,

1	12
100	
48	5
	5

Product of the larger set, 4800. Do. of the less set, 300. Quotient, 16.

the interest varies directly as the principal, we get the proportion,
$100 : 16 \times 12 :: 5 : x$,
whence $x = \frac{16 \times 12 \times 5}{100} = 9\frac{3}{5}$,
and the reason for the rule in § 79 is evident. Similarly for the other parts of the example.]

81. Example. If the interest of a hundred for a month and one-third be five and one-fifth, say what the interest is of sixty-two and a half for three months and one-fifth.

Statement: $\begin{array}{ccc} \frac{4}{3} & & \frac{16}{5} \\ 100 & & \frac{125}{2} \\ \frac{26}{5} & & \end{array}$ Answer[1]: interest $7\frac{4}{5}$.

[This may be worked out similarly as the preceding example.]

82. Example of the Rule of Seven. If eight best variegated silk scarfs, measuring three cubits in breadth and eight in length) cost a hundred (*nishkas*); say quickly, merchant, if thou understand trade, what a like scarf, three and a half cubits long and half a cubit wide, will cost.

---

[1] Transposing the fruit, $\begin{array}{cc} \frac{4}{3} & \frac{16}{5} \\ 100 & \frac{125}{2} \\ & \frac{26}{5} \end{array}$ and the denominators, $\begin{array}{cc} 4 & 16 \\ 5 & 3 \\ 100 & 125 \\ 2 & \\ 5 & 26 \end{array}$

Abridging by correspondent reduction on both sides, $\begin{array}{cc} 1 & 4 \\ 5 & 3 \\ 4 & 5 \\ 2 & \\ 5 & 26 \end{array}$ and by

further reduction, $\begin{array}{cc} 1 & 1 \\ 1 & 3 \\ 1 & 1 \\ 1 & \\ 5 & 13 \end{array}$

Product of the larger set, 39.
Do. of the less set, 5. Quotient, $7\frac{4}{5}$.

The abridgment of the work by reduction of terms on both sides by their common divisors is taught by the *Manoranjana*.

Statement:

3	$\frac{1}{2}$
8	$\frac{7}{2}$
8	1
100	

Answer[1]: *Nishka* 0, *drammas* 14, *panas*, 9, *kákiní* 1, cowry-shells $6\frac{2}{3}$.

[Let x denote the cost required.

Then the area of cloth in the first case

$= 8 \times 3 \times 8$ sq. cubits;

and in the second case

$= 1 \times \frac{7}{2} \times \frac{1}{2}$ sq. cubits.

Hence, the quality of the cloth remaining the same, we get the proportion,

$$8 \times 3 \times 8 : 1 \times \frac{7}{2} \times \frac{1}{2} :: 100 : x,$$

whence $x = \frac{1 \times 7 \times 1 \times 100}{8 \times 3 \times 8 \times 2 \times 2}$,

and the reason for the process given in the foot-note is evident.]

83. Example of the Rule of Nine. If thirty benches, twelve fingers thick, square of four wide, and fourteen cubits long, cost a hundred (*nishkas*); tell me, my friend, what price fourteen benches will fetch, which are four less in every dimension.

Statement:

12	8
16	12
14	10
30	14
100	

Answer[2]: *nishkas* $16\frac{2}{3}$.

---

[1] Transposing fruit and denominators,

3	1
2	
8	7
2	
8	1
	100

Product of the larger set, 700.
Do. of the less set, 768. Quotient: 0, 14, 9, 1, $6\frac{2}{3}$.

[2] Transposing fruit,

12	8
16	12
14	10
30	14
	100

Abridging by correspondent reduction on both sides,

1	1
2	1
1	1
3	1
	100

Product of the larger set, 100.
Do. of the less set, 6. Quotient, $16\frac{2}{3}$.

[Here, putting x for the price, and proceeding as above, we get the proportion,
$30 \times 12 \times 16 \times 14 \times 24 : 14 \times 8 \times 12 \times 10 \times 24 :: 100 : x$, whence x is known.]

84. Example of the Rule of Eleven. If the hire of carts to convey the benches of the dimensions first specified (in the preceding example), through a distance of one league (*gavyúti*[1]) be eight *drammas*; say what the cart-hire should be for bringing the benches last mentioned, four less in every dimension, through a distance of six leagues.

Statement:			
	12	8	
	16	12	Answer[2]: *drammas* 8.
	14	10	
	30	14	
	1	6	
	8		

[Let x denote the cart-hire required.

Then in the first case, solid content of benches $= 30 \times 12 \times 16 \times 14 \times 24$ cubic fingers, and they are carried through a distance of 1 league; and in the second case, solid content of benches $= 14 \times 8 \times 12 \times 10 \times 24$ cubic fingers, and they are carried through a distance of 6 leagues, which is the same as a solid content of $14 \times 8 \times 12 \times 10 \times 24 \times 6$ cubic fingers carried through 1 league;

---

[1] *Garyúti*, two *krosas* or half a *yojana*; it contains rather more than 8000 yards, and is more than 4¼ English miles.

[2] Transposing the fruit,

12	8
16	12
14	10
30	14
1	6
	8

Abridging by correspondent reduction on both sides,

1	1
2	1
1	1
3	1
1	6
	8

and by further reduction,

1	1
1	1
1	1
1	1
1	2
	4

Product of the larger set, 8.
Do. of the less set, 1.
Quotient, 8.

hence, since for a given distance, the hire will vary directly as the solid content of benches carried, we get the proportion, $30 \times 12 \times 16 \times 14 \times 24 \times 1 : 14 \times 8 \times 12 \times 10 \times 24 \times 6 ::$ $8 : x$,

whence $x = \frac{14 \times 8 \times 12 \times 10 \times 6 \times 8}{30 \times 12 \times 16 \times 14} = 8$.]

85. Rule of barter[1]; half a stanza. So in barter likewise, the same process is (followed); transposing both prices, as well as the divisors.[2]

[The reason for the rule will appear from the example which follows.]

86. Example. If three hundred mangoes be had in the market for one *dramma*, and thirty ripe pomegranates for a *pana*; say quickly, friend, how many should be had in exchange for ten mangoes.

Statement: 16 1 Answer[3]: 16 pomegranates.

16	1
300	30
10	

---

[1] *Bhánda-prati-bhándaka*, commodity for commodity; computation of the exchange of goods (*vastu-vinimaya-ganita*,—Gang.); barter.

[2] Gangádhara, Súryadása and the *Manoranjana*, so read this passage, *haránscha-múlye*. But Ganesa and Ranganátha have the affirmative adverb *sadáhi* (always) in place of the word *haránscha* (and the divisors). At all events, the transposition of denominators takes place, as usual; and so does that of the lowest term or fruit, as in the Rule of Five, to which, as Súryadása remarks, this is analogous. It comprises two proportions, thus stated by him from the example in the text:—"If for one *pana*, thirty pomegranates may be had, how many for sixteen? Answer, 480. Again, if for 300 mangoes, 480 pomegranates may be had, how many for ten? Answer, 16. Here thirty is first multiplied by sixteen and then divided by one; and then multiplied by ten and divided by three hundred. For brevity, the prices are transposed, and the result is the same."

[3] Transposing the prices,

1	16
300	30
10	

and transposing the fruit,

1	16
300	30
	10

Product of the larger set, 4800. Quotient, 16.
Do. of the less set, 300.

[The example involves two proportions, as Súryadása observes. First find how many pomegranates can be had for one *dramma* or 16 *panas*.

Putting y for this number, we get

$$30 : y :: 1 : 16,$$

$$\therefore y = 30 \times 16.$$

Hence by the question, 300 mangoes are equivalent to 30×16 pomegranates.

Then, putting x for the number of pomegranates required, we get the proportion,

$$300 : 10 :: 30 \times 16 : x,$$

$$\text{whence } x = \frac{30 \times 16 \times 10}{300} = 16,$$

and the reason for the rule in § 85 is obvious.]

CHAPTER IV.
INVESTIGATION OF MIXTURE.[1]

SECTION I.
INTEREST.

87—88. Rule : a stanza and a half.[2]

The argument[3] multiplied by its time, and the fruit multiplied by the mixed quantity's time, being severally set down, and divided by their sum and multiplied by the mixed quantity, are the principal and interest (composing the quantity). Or the principal being found by the rule of supposition (§ 50), that, taken from the mixed quantity, leaves the amount of interest.

[The rule refers specially to the example given in § 89. By the word *argument* is meant 100, and by the word *fruit* is meant the interest on 100 for 1 month, or, as we would call it, the rate per cent. per mensem.

Let r = rate per cent. per mensem.
t = time in years.
P = principal.
I = interest.
A = amount.

[1] *Misra-vyavahára*, investigation of mixture, ascertainment of composition, as principal and interest, and so forth.—Gan. It is chiefly grounded on the rule of proportion.—*Ibid*. The rules in this chapter bear reference to the examples which follow them. Generally they are *quæstiones otiosæ*, problems for exercise.

[2] To investigate the principal and interest, the amount, time and rate being given.—Gan.

[3] *Pramána*, argument; *phala*, fruit (§ 70): principal and interest.

Then, $I = \frac{P \times r \times 12 \times t}{100}$.

$$\therefore A = P\left\{1 + \frac{r \times 12 \times t}{100}\right\} = P \times \frac{100 + r \times 12 \times t}{100}$$

$$\therefore P = \frac{A \times 100}{100 + r \times 12 \times t}, \text{ and } I = \frac{A \times r \times 12 \times t}{100 + r \times 12 \times t}.$$

The last two formulæ stated in words give the first part of the rule. It is evident that these formulæ may also be derived by a simple proportion, as observed by Ganesa.

In the latter part of the rule, the principal is found by the author's method of supposition, which is practically the same as the solution of a simple equation.]

89. Example. If the principal sum, with interest at the rate of five on the hundred by the month, amount in a year to one thousand, tell the principal and int rest respectively.

Statement: 1 12
100 1000.
5

Answer[1]: principal, 625 ; interest, 375.

Or, by the rule of position, put one; and proceeding according to that rule (§ 50), the interest of unity is $\frac{3}{5}$, which, added to one, makes $\frac{8}{5}$. The given quantity 1000, multiplied by unity and divided by that ($\frac{8}{5}$), shows the principal 625. This, taken from the mixed amount, leaves the interest[2] 375.

[The second solution follows the latter part of the preceding rule.

[1] 100 multiplied by 1 is 100; 5 by 12 is 60. Their sum 160 is the divisor. The first number 100, multiplied by 1000, and divided by 160, is 625. The second 60, multiplied by 1000, and divided by 160, gives 375.—Gang.

[2] Or the principal being known, the interest may be found by the Rule of Five.—Súr.

Let x denote the principal required. Then ∵ the rate per cent. per annum is 60, we get

$$x\left(1+\frac{60}{100}\right)=1000,$$

$$\text{whence } x=\frac{1000}{1+\frac{3}{5}}.\ ]$$

90. Rule[1]: The arguments taken into their respective times are divided by the fruit taken into the elapsed times; the several quotients, divided by their sum and multiplied by the mixed quantity, are the parts as severally lent.

[The rule refers specially to the example in § 91.

Let x, y, z, be the portions lent at r_1, r_2, r_3 per cent. per mensem, and let I = common interest in t_1, t_2, t_3 months respectively.

Then, $x+y+z=a$, a given quantity;

$$\text{and } \frac{x\times r_1\times t_1}{100}=\frac{y\times r_2\times t_2}{100}=\frac{z\times r_3\times t_3}{100}=I;$$

$$\therefore x:y:z::\frac{100\times 1}{r_1\times t_1}:\frac{100\times 1}{r_2\times t_2}:\frac{100\times 1}{r_3\times t_3};$$

$$\therefore x=\frac{100\times 1}{r_1\times t_1}\times\frac{a}{\frac{100\times 1}{r_1\times t_1}+\frac{100\times 1}{r_2\times t_2}+\frac{100\times 1}{r_3\times t_3}},$$

with similar values for y and z. Hence the rule. Ganesa's explanation is practically the same as the above, but it is rather obscure: see footnote 2.]

91. Example. The sum of six less than a hundred *nishkas* being lent in three portions at interest of five, three and four per cent., an equal interest was obtained on all three portions, in seven, ten and five months respectively. Tell, mathematician, the amount of each portion.[2]

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[1] For determining parts of a compound sum.—Súr.

[2] Since the amount of interest on all the portions is the same, put unity for its arbitrarily assumed amount: whence corresponding principal sums

Statement: $\begin{matrix} 1 & 7 \\ 100 & \\ 5 & \end{matrix} \quad \begin{matrix} 1 & 10 \\ 100 & \\ 3 & \end{matrix} \quad \begin{matrix} 1 & 5 \\ 100 & \\ 4 & \end{matrix}$

Mixed amount 94.

Answer[1]: the portions are 24, 28 and 42. The equal amount of interest is $8\frac{2}{5}$.

92. Rule[2]: half a stanza.

The contributions[3] being multiplied by the mixed amount and divided by the sum of the contributions, are the respective fruits.[4]

[If we divide the mixed amount into parts proportional to the contributions, we shall get the mixed amount as due to each contribution. And this amount less the contribution is the gain. Hence the reason for the rule is obvious.]

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are found by the Rule of Five. For instance, if a hundred be the capital, of which five is the interest for a month, what is the capital, of which unity is the interest for seven months? and in like manner, the other principal sums are to be found. Thus, a compound proportion being wrought, the time is multiplied by the argument to which it appertains, and divided by the fruit taken into the elapsed time. Then, as the total of those principal sums is to them severally, so is the given total to the respective portions lent. They are thus severally found by the Rule of Three.—Gan.

[1] Multiplying the argument and fruit by the times, and dividing one product by the other, there result the fractions $\frac{100}{35}$, $\frac{100}{30}$, $\frac{100}{20}$, or $\frac{20}{7}$, $\frac{20}{6}$, $\frac{20}{4}$; which reduced to a common denominator and summed, make $\frac{1880}{168}$. Multiplied by the mixed amount 94, the fractions become $\frac{1880}{7}$, $\frac{1880}{6}$, $\frac{1880}{4}$; and then divided by the sum $\frac{1880}{168}$, they give $\frac{168}{7}$, $\frac{168}{6}$, $\frac{168}{4}$, or 24, 28, 42.—*Mano.*

To find the interest, employ the Rule of Five; $\begin{matrix} 1 & 7 \\ 100 & 24 \\ 5 & \end{matrix}$. Answer, $8\frac{2}{5}$. By the same method, with all three portions, the interest comes out the same.—Súr.

[2] The capital sums, their aggregate amount, and the sum of the gains being given, to apportion the gains.—Gan.

[3] *Prakshepaka*, that which is thrown in or mixed.—Gan. Joined together.—Súr.

[4] The principle of the rule is obvious, being simply the Rule of Three.—Gan.

If by this sum of contributions, this contribution be had, then by the compound sum what will be? The numbers thus found, less the contributions, are the gains.—Ranganátha on the *Vásaná*.

93. Example. Say, mathematician, what the apportioned shares are of three traders, whose original capitals were respectively fifty-one, sixty-eight and eighty-five, which have been raised by commerce conducted by them on joint stock, to the aggregate amount of three hundred.

Statement: 51, 68, 85; sum, 204. Mixed amount, 300.

Answer: 75, 100, 125. These, less the capital sums, are the gains: *viz.*, 24, 32, 40.

Or the mixed amount, less the sum of the capitals, is the profit on the whole: *viz.*, 96. This being multiplied by the contributions and divided by their sum, gives the respective gains: *viz.*, 24, 32, 40.

[The whole gain being divided into parts proportional to the contributions, gives the respective gains.]

Section II.

94. Rule[1]: half a stanza Divide denominators by numerators; and then divide unity by those quotients added together. The result will be the time of filling (a cistern by several fountains).

[The rule refers to an example of the class given in § 95.

Let the times in which the fountains can severally fill the cistern be $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, &c., of a day. Then in one day, the parts of the cistern filled up by the fountains severally are $\frac{b_1}{a_1}$, $\frac{b_2}{a_2}$, &c.; $\therefore$ if they work together, the part filled in one day will be $=\frac{b_1}{a_1}+\frac{b_2}{a_2}+$&c.;

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[1] To apportion the time for a mixture of springs to fill a well or cistern. —Gan. To solve an instance relative to fractions.—Súr.

∴ the time in which the whole cistern will be filled $= \dfrac{1}{\dfrac{b_1}{a_1}+\dfrac{b_2}{a_2}+\&c.}$ of a day,

whence the reason for the rule is evident. The explanations of this rule given by the commentators are all vague and unsatisfactory.]

95. Example. Say quickly, friend, in what portion of a day (four) fountains, being let loose together, will fill a cistern, which, if severally opened, they would fill in one day, half a day, the third, and the sixth part, respectively.

Statement: $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$.

Answer: $\frac{1}{12}$th part of a day.

Section III.

Purchase and Sale.

96. Rule.[1] By the (measure of the) commodities,[2] divide their prices taken into their respective portions (of the purchase); and by the sum of the quotients divide both them and those portions severally multiplied by the mixed sum: the prices and quantities are found in their order.

[The reason for the rule will appear from the solution of the example in § 97.]

97. Example. If three and a half *mánas*[3] of rice may be had for one *dramma*, and eight of kidney-beans[4]

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[1] For a case where a mixture of portions and composition of things are given.—Gan. Concerning measure of grain, &c.—Súr.

[2] *Panya*, the measure of the grain or other commodity procurable for the current price in the market.—Súr, and the *Mana*.

[3] *Mána* or *mánaka*, a measure; seemingly intending a particular one. According to Ganesa, the *mana* (apparently the same as the *mána*) is at most an eighth of a *khári*; being a cubic span. See note to § 236. A spurious couplet (see note on § 2) makes it the modern measure of weight containing forty *sers*.

[4] *Mudga*, phaseolus mungo; a sort of kidney-bean.

for the like price, take these thirteen *kákinís*, merchant, and give me quickly two parts of rice with one of kidney-beans; for we must make a hasty meal and depart, since my companion will proceed onwards.

Statement: $\frac{2}{1}$ $\frac{1}{1}$. Mixed sum $\frac{13}{64}$.
$\frac{1}{1}$ $\frac{1}{1}$
$\frac{7}{2}$ $\frac{8}{1}$

The prices, $\frac{1}{1}$, $\frac{1}{1}$, multiplied by the portions $\frac{2}{1}$, $\frac{1}{1}$, and divided by the goods $\frac{7}{2}$, $\frac{8}{1}$, make $\frac{4}{7}$, $\frac{1}{8}$, the sum of which is $\frac{39}{56}$. By this divide the same fractions $\frac{4}{7}$, $\frac{1}{8}$, taken into the mixed sum $\frac{13}{64}$; and the portions $\frac{2}{1}$, $\frac{1}{1}$, taken into that mixed sum $\frac{13}{64}$. There result the prices of the rice and kidney-beans, $\frac{1}{6}$ and $\frac{7}{192}$ of a *dramma*; or 10 *kákinís* and $13\frac{1}{3}$ shells for the rice, and 2 *kákinís* and $6\frac{2}{3}$ shells for the kidney-beans; and the quantities are $\frac{7}{12}$ and $\frac{7}{24}$ of a *mána* of rice and kidney-beans respectively.

[Let x denote the *mánas* of kidney-beans.
Then $2x$ will denote the *mánas* of rice.
Now the price paid $= \frac{13}{64}$ of a *dramma*;
$\therefore 2x \times \frac{2}{7} + x \times \frac{1}{8} = \frac{13}{64}$;
Whence $x \left(\frac{4}{7} + \frac{1}{8}\right) = \frac{13}{64}$,

$$\therefore x = \frac{1 \times \frac{13}{64}}{\frac{4}{7} + \frac{1}{8}} = \frac{7}{24},$$

$$\text{and } 2x = \frac{2 \times \frac{13}{64}}{\frac{4}{7} + \frac{1}{8}} = \frac{7}{12};$$

and the prices are, $\frac{7}{24} \times \frac{1}{8} = \frac{7}{192}$
and $\frac{7}{24} \times \frac{4}{7} = \frac{1}{6}$.

Hence the reason for the rule is evident. General formulæ corresponding to the rule in § 96 may be easily established. It is, however, not worth while to do so.

The example in § 98 may be worked out in a similar manner.]

98. Example. If a *pala* of best camphor may be had for two *nishkas*, and a *pala* of sandal-wood[1] for the eighth part of a *dramma*, and half a *pala* of alœ-wood[2] also for the eighth of a *dramma*, good merchant, give me the value of one *nishka* in the proportions of one, sixteen and eight; for I wish to prepare a perfume.

Statement: 32 $\frac{1}{8}$ $\frac{1}{8}$. Mixed sum 16.
1 1 $\frac{1}{2}$
1 16 8

Answer. Prices: *drammas*, $14\frac{2}{9}$, $\frac{8}{9}$, $\frac{8}{9}$.
Quantities: *palas*, $\frac{4}{9}$, $7\frac{1}{9}$, $3\frac{5}{9}$.

SECTION IV.

99. Rule. Problem concerning a present of gems.[3] From the gems subtract the gift multiplied by the persons; and any arbitrary number being divided by the remainders, the quotients are numbers expressive of the prices. Or the remainders being multiplied together, the product, divided by the several reserved remainders, gives the values in whole numbers.

[The reason for the rule will appear from the solution of the example in § 100, to which the rule specially refers.]

100. Example. Four jewellers, possessing respectively eight rubies, ten sapphires, a hundred pearls, and five diamonds, presented, each from his own stock, one apiece to the rest in token of regard and gratification at meeting; and they thus became owners of stock of precisely equal value. Tell me severally, friend, the prices of their gems.

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[1] *Chandana*; santalum album.
[2] *Aguru*; aquillaria agallochum.
[3] The problem is an indeterminate one. The solution gives relative values only.

Statement : rub. 8 ; sapph. 10 ; pearls 100 ; diam. 5 ; gift 1 ; persons 4.

Here, the product of the gift 1 by the persons 4, *viz.*, 4, being severally subtracted, there remain rubies 4, sapphires 6, pearls 96, diamond 1. Any number arbitrarily assumed being divided by these remainders, the quotients are the relative values. Taking it at random, they may be fractional values ; or by judicious selection, whole numbers Thus, put 96 ; and the prices thence deduced are 24, 16, 1, 96 ; and the equal stock 233.

Or the remainders being multiplied together, and the product severally divided by those remainders, the prices are 576, 384, 24, 2304 ; and the equal amount of stock (after interchange of presents) is 5592.

[Let the relative values of a ruby, sapphire, pearl, and diamond be respectively x, y, z, w. Then we shall evidently get from the conditions of the problem the following equations :—

$$\begin{aligned}&5x+y+z+w\\&=7y+x+z+w\\&=97z+x+y+w\\&=2w+x+y+z\end{aligned}$$

$\therefore 4x=6y=96z=w=k$ suppose ; then $x=\frac{k}{4}, y=\frac{k}{6}, z=\frac{k}{96}, w=k.$

Putting $k=$ L. C. M. of 4, 6, 96, *i.e.*, 96, we get the least integral values of x, y, z, w, *viz.*, 24, 16, 1, 96 ; and putting k = product of 4, 6, 96, we get for x, y, z, w, the values 576, 384, 24, 2304.

The reason for the rule in § 99 will be evident from the above algebraical solution. The coefficients of x, y, &c. in the final equations will be

= the no. of the respective gems—the no. of that gem given to each person × the no. of persons altogether.

For instance, if there were 9 rubies, and each presented 2 from his pocket to each of the rest, the coefficient of x in the left hand side of the first equation would have been 3; and that of x in the right hand side would have been 2; thus the coefficient of x in the simplified equations would have been 1, *i.e.*, $9 - 4 \times 2$, which agrees with the rule in § 99.

Súryadása cites the *Víja-ganita* for the solution of the problem. Ranganátha gives an arithmetical explanation which, however, is meaningless add obscure.]

SECTION V.

ALLIGATION.[1]

101. Rule.[2] The sum of the products of the touch[3] and (weight of several parcels)[4] of gold being divided by the aggregate of the gold, the touch of the mass is found: or (after refining) being divided by the fine gold, the touch is ascertained; or divided by the touch, the quantity of purified gold is determined.

[This is simply the ordinary rule for alligation medial. We may consider the prices per *másha* of the several kinds of gold as *proportional to* the fineness. The reason for the rule is obvious.]

102—103. Example. Parcels of gold weighing severally ten, four, two and four *máshas*, and of the fineness of thirteen, twelve, eleven and ten respectively, being melted together, tell me quickly, merchant, who art conversant with the computation of gold, what the fineness of the mass is. If the twenty *máshas* above

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[1] *Suvarna-ganita*, computation of gold, that is, of its weight and fineness; alligation medial.

[2] To find the fineness produced by mixture of parcels of gold; and, after refining, to find the weight, if the fineness be known; and the fineness, if the weight of refined gold be given.—Gan.

[3] *Varna*, colour of gold on the touchstone; fineness of gold determined by that touch. See § 77. "The degrees of fineness increase as the weight is reduced by refining."—Gan.

[4] Gang.

described be reduced to sixteen by refining, tell me instantly the touch of the purified mass. Or, if its purity when refined be sixteen, prithee, what is the number to which the twenty *máshas* are reduced ?

Statement : touch 13 12 11 10 ;
weight 10 4 2 4.

Answer[1]: after melting, fineness 12 ; weight 20.

After refining, the weight being sixteen *máshas*, the touch is 15. The touch being sixteen, the weight is 15.

104. Rule.[2] From the acquired fineness of the mixture, taken into the aggregate quantity of gold, subtract the sum of the products of the weight and fineness (of the parcels, the touch of which is known), and divide the remainder by the quantity of gold of unknown fineness ; the quotient is the degree of its touch.[3]

105. Example. Eight *máshas* of ten, and two of eleven by the touch, and six of unknown fineness, being mixed together, the mass of gold, my friend, became of the fineness of twelve ; tell the degree of unknown fineness.

Statement : 10 11 Fineness of the mixture 12.
8 2 6 .

Answer : degree of the unknown fineness 15.

[Let x denote the unknown fineness. Then, $(8+2+6)\times 12 = 8\times 10+2\times 11+6\times x$,

whence $x=\dfrac{(8+2+6)\times 12-(8\times 10+2\times 11)}{6}$. The reason for the rule in § 104 is obvious.]

[1] Products 130, 48, 22, 40. Their sum 240, divided by 20, gives 12 ; divided by 16, gives 15.

[2] To discover the fineness of a parcel of unknown degree of purity mixed with others of which the touch is known.—Gan.

[3] The rule being the converse of the preceding, the principle of it is obvious.—Rang.

106. Rule.[1] The acquired fineness of the mixture being multiplied by the sum of the gold (in the known parcels), subtract therefrom the aggregate products of the weight and fineness (of the parcels): divide the remainder by the difference between the fineness of the gold of unknown weight and that of the mixture, the quotient is the weight of gold that was unknown.

107. Example. Three *máshas* of gold of the touch of ten, and one of the fineness of fourteen, being mixed with some gold of the fineness of sixteen, the degree of purity of the mixture, my friend, is twelve. How many *máshas* are there of the fineness of sixteen?

Statement: 10 14 16. Fineness of the mixture 12.
3 1

Answer: *másha* 1.

[Let x denote the number of *máshas* required.
Then, $(3+1+x)\times 12 = 3\times 10 + 1\times 14 + x\times 16$;

$$\therefore x = \frac{(3+1)\times 12 - (3\times 10 + 1\times 14)}{16-12},$$

whence the rule in § 106.]

108. Rule.[2] Subtract the effected fineness from that of the gold of a higher degree of touch, and that of the one of lower touch from the effected fineness; the differences, multiplied by an arbitrarily assumed number, will be the weights of gold of the lower and higher degrees of purity respectively.

109. Example. Two ingots of gold, of the touch of sixteen and ten respectively, being mixed together,

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[1] To find the weight of a parcel of known fineness, but unknown weight, mixed with other parcels of known weight and fineness.—Gan.

[2] To find the weight of two parcels of given fineness and unknown weight.—Gan. and Súr. The problem is an indeterminate one, as is intimated by the author.

the gold became of the fineness of twelve. Tell me, friend, the weight of gold in both lumps.

Statement: 16, 10. Fineness resulting 12.

Putting one, and proceeding as directed, the weights of gold are found, *máshas* 2 and 4. Assuming two, they are 4 and 8. Taking half, they come out 1 and 2. Thus, manifold answers are obtained by varying the assumption.

[Let x and y be the weights required.

Then, $x \times 16 + y \times 10 = (x + y) \times 12$;

$\therefore (16-12)\, x \quad = (12-10) \times y$;

$\therefore \quad \frac{x}{y} \quad = \frac{12-10}{16-12}$;

$\therefore x = (12 - 10)\, k,\ y = (16-12)\, k$,

where k is any positive quantity.

The general solution in positive integers evidently is, $x = k$, $y = 2k$, where k is any positive integer.

The reason for the rule in § 108 is obvious.]

SECTION VI.

PERMUTATIONS AND COMBINATIONS.

110—112. Rule[1]: three stanzas.

Let the figures from one upwards, differing by one, put in the inverse order, be divided by the same (arithmeticals) in direct order; and let the subsequent be multiplied by the preceding, and the next following by the foregoing (result). The several results are the changes, ones, twos, threes, &c.[2] This is termed a

[1] To find the possible permutations of long and short syllables in prosody; combinations of ingredients in pharmacy; variations of notes, &c., in music; as well as changes in other instances.—Gan.

[2] According to Gaṇesa, there is no demonstration of the rule, besides acceptation and experience. [This, however, is not correct.—Ed.]

general rule.[1] It serves in prosody, for those versed therein, to find the variations of metre; in the arts (as in architecture) to compute the changes upon apertures (of a building); and (in music) the scheme of musical permutations[2]; in medicine, the combinations of different savours. For fear of prolixity, this is not (fully) set forth.

[The reason for the rule will appear from the solution of the example which follows.]

113. A single example in prosody. In the permutations of the *Gáyatrí* metre,[3] say quickly, friend, how many the possible changes of the verse are; and tell severally, how many the permutations are with one, (two, three,) &c., long syllables.

Here the verse of the *Gáyatrí* stanza comprises six syllables. Wherefore, the figures from one to six are set down, and the statement of them, in direct and inverse order is $\begin{matrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$. Proceeding as directed, the results are :—changes with one long syllable, 6; with two, 15; with three, 20; with four, 15; with five, 6; with six, 1; with all short, 1. The sum of these is the whole number of permutations of the verse, 64.

[1] Commentators appear to interpret this as a name of the rule here taught; *sádhárana*, or *sádhárana-chhandoganita*, general rule of prosodian permutation, subject to modification in particular instances, as in music, where a special method (*asádhárana*) must be applied.—Gang. and Súr.

[2] *Khanda-meru*, a certain scheme.—Gan. It is more fully explained by other commentators; but the translator is not sufficiently conversant with the theory of music to understand the term distinctly.

[3] The *Gáyatrí* metre in sacred prosody is a triplet comprising twenty-four syllables, as in the famous prayer containing the Brahmanical creed, called *Gáyatrí* [*Rigveda*, *Mandala* 3, *súkta* 62, *rik* 10.—Ed.] See As. Res., vol. X, p. 463. But in the prosody of profane poetry, the same number of syllables is distributed in a tetrastic; and the verse consequently contains six syllables. (As. Res., vol. X, p. 469.)

In like manner, setting down the numbers of the whole tetrastic, in the mode directed, and finding the changes with one, two, &c., and summing them, the permutations of the entire stanza are found, *viz.*, 16777216.

In the same way may be found the permutations of all varieties of metre, from *Ukthá* (which consists of monosyllabic verses) to *Utkriti* (the verses of which contain twenty-six syllables).[1]

[In the *Gáyatrí* metre, the number of syllables is 6. In finding the number of changes with *one* long and the rest *five* short syllables, we have to find the permutations of 6 things taken all at a time, when one of them is of one kind, and the rest, of another kind. Hence the number of changes $= \dfrac{\underline{|6}}{\underline{|1}\ \underline{|5}}$, which is precisely the number of combinations of 6 things taken 1 at a time. Similarly, in finding the number of changes with *two* long, and therefore the rest *four* short, we get the number $= \dfrac{\underline{|6}}{\underline{|2}\ \underline{|4}}$, and so on; thus finally the total number of changes = sum of combinations of 6 things taken 1, 2, 3, 4, 5, 6, at a time + 1 (with all short syllables) $= (2^6 - 1) + 1 = 64$.

Now $\dfrac{\underline{|6}}{\underline{|1}\ \underline{|5}} = \dfrac{6}{1}$; $\dfrac{\underline{|6}}{\underline{|2}\ \underline{|4}} = \dfrac{6 \times 5}{1 \times 2}$; &c.

Hence the reason for the rule is clear.

If the aggregate number of changes only is wanted, this can be found at once from the proposition, *viz.*, the total number of combinations of n things taken 1, 2, 3......n at a time $= 2^n - 1$ (see Todhunter's Algebra, art. 515). This proposition is given in a concrete shape in § 130 – 131.

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[1] As. Res., vol. X, pp. 468—473.

Similarly, taking the whole tetrastic, *i.e.*, 24 syllables, the total number of changes = $(2^{24}-1)+1$ (with all short syllables) = 16777216.]

114. Example. In a pleasant, spacious and elegant edifice, with eight doors,[1] constructed by a skilful architect, as a palace for the lord of the land, tell me the permutations of apertures taken one, two, three, &c.[2] Say, mathematician, how many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt and bitter,[3] taking them by ones, twos, threes, &c.

Statement, first example:

8 7 6 5 4 3 2 1
1 2 3 4 5 6 7 8 .

Answer: the number of ways in which the doors may be opened by ones, twos, threes, &c., is 8, 28, 56, 70, 56, 28, 8, 1, respectively. And the changes on the apertures of the octagonal palace[4] amount to 255.

Statement, second example:

6 5 4 3 2 1
1 2 3 4 5 6 .

Answer: the number of various preparations with ingredients of divers tastes is 6, 15, 20, 15, 6, 1.[5]

[In the first example, the total number of variations $= 2^8 - 1 = 255$. The case of all the windows being shut is not taken

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[1] *Múshá*, aperture for the admission of air; a door or window; (same with *gaváksha.*—Gan.) A portico or terrace, (*bhúmi-visesha.*—Gang. and Súr.)

[2] The variations of one window or portico open (or terrace unroofed) and the rest closed; two open, and the rest shut; and so forth.

[3] *Amara-kosha*, *swarga-varga*, 147.

[4] An octagonal building, with eight doors or windows or porticos or terraces facing the eight cardinal points of the horizon, is meant —Gan.

[5] Total number of possible combinations is 63.—Gang.

into account; otherwise the total number of variations would be 256.

The second example from its very nature is a case of combinations and not of permutations, *i.e.*, we have to find the number of combinations of 6 things taken 1, 2......6 at a time. The rule in §110—112, however, equally applies, as has been explained above. The total number of combinations in this case $= 2^6 - 1 = 63$.]

CHAPTER V.
PROGRESSIONS.[1]

SECTION I.
ARITHMETICAL PROGRESSION.

115. Rule.[2] Half the period[3] multiplied by the period added to unity, is the sum of the arithmeticals one, &c., and is named their addition.[4] This, being multiplied by the period added to two, and being divided by three, is the aggregate of the additions.[5]

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[1] *Sredhí*, a term employed by the older authors for any set of distinct substances or other things put together.—Gan. It signifies sequence or progression. *Sredhí-vyavahára*, ascertainment or determination of progressions.

[2] To find the sums of the arithmeticals. Gan.

[3] *Pada*, the place.—Gan. Any one of the figures or digits, being that of which the sum is required.—Súr. The last of the numbers to be summed.—*Mano.* See below, note to §119.

[4] *Sankalita*, the first sum or addition of arithmeticals. *Sankalitaikya*, aggregate of additions, summed sums or second sum.

[5] The first figure is unity. The sum of that and the period being halved, is the middle figure. As the figures decrease behind it, so they increase before it: wherefore the middle figure, multiplied by the period, is the sum of the figures one, &c., continued to the period. The only proof of the rule for the aggregate of sums is acceptation.—Gan. [This last remark is not correct. The proof of the formula for the sum of the first n arithmeticals given by Ganesa does not apply where n is even, but requires to be modified. In that case, the sum of any two terms equidistant from the first and last $= n+1$; whence the sum of n terms evidently $= \frac{n}{2}(n+1)$.—Ed.] It is a maxim, that a number multiplied by the next following arithmetical, and halved, gives the sum of the preceding; wherefore, &c.—Súr. Kamalákara is quoted by Ranganátha for a demonstration grounded on placing the numbers of the series in the reversed order under the direct one and adding the two series—[the same proof as that given in modern works on Algebra.—Ed.]

[$1+2+3......$to n terms $=\frac{n(n+1)}{2}$. By "the aggregate of the additions," the author evidently means the sum of n terms of the series whose nth term is $\frac{1}{2}n(n+1)$, in other words, the sum of *n triangular numbers*. This sum is $\frac{1}{6}n(n+1)(n+2)$ $=\frac{n(n+1)}{2}\cdot\frac{n+2}{3}$. See Todhunter's Algebra, Art. 666. The reason for the rule is obvious.]

116. Example. Tell me quickly, mathematician, the sums of the several (progressions of) numbers one, &c., continued to nine; and the summed sums of those numbers.

Statement: arithmeticals: 1 2 3 4 5 6 7 8 9.

Answer: sums: 1 3 6 10 15 21 28 36 45.

Summed sums: 1 4 10 20 35 56 84 120 165.

117. Rule.[1] Twice the period added to one and divided by three, being multiplied by the sum (of the arithmeticals), is the sum of the squares. The sum of the cubes of the numbers one, &c., is pronounced by the ancients equal to the square of the addition.

[$1^2+2^2+......+n^2=\frac{n(n+1)(2n+1)}{6}$

$$=\frac{n(n+1)}{2}\cdot\frac{2n+1}{3}.$$

$1^3+2^3+......+n^3=\left\{\frac{n(n+1)}{2}\right\}^2$. See Todhunter's Algebra, Arts. 460, 461.]

118. Example. Tell promptly the sum of the squares, and the sum of the cubes, of those numbers, if thy mind be conversant with the way of summation.

Statement: 1 2 3 4 5 6 7 8 9.

Answer: sum of squares, 285. Sum of cubes, 2025.

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[1] To find the sums of squares and of cubes.—Gan. and Súr.

119. Rule.[1] The increase multiplied by the period less one, and added to the first quantity, is the amount of the last.[2] That, added to the first, and halved, is the amount of the mean; which multiplied by the period is the amount of the whole, and is denominated (*ganita*) the computed sum.

[Consider the series a, $a+b$, $a+2b$......The nth term $=a+(n-1)b$. If n be *odd*, middle term $=\frac{n+1}{2}$ th term $=\frac{2a+(n-1)b}{2}$. If n be *even*, there will be two middle terms, *viz.*, $\frac{n}{2}$ th and $\frac{n+2}{2}$ th terms, and the mean amount = average of these two terms $=\frac{n}{2}$ th term $+\frac{1}{2}$ of common difference $=a+\left(\frac{n}{2}-1\right)b+\frac{1}{2}b=\frac{2a+(n-1)b}{2}$. And the sum of n terms $=\frac{n}{2}\left\{2a+(n-1)b\right\}$.

Thus the rule holds good whether n be odd or even. The author does not notice that the first part of the rule in § 115 is only a particular case of the present rule.]

120. Example. A person, having given four *drammas* to priests on the first day, proceeded, my friend, to distribute daily alms at a rate increasing by five a

[1] Where the increase is arbitrary.—Gang. In such cases, to find the last term, mean amount, and sum of the progression.—Súr. From first term, common difference and period, to find the whole amount, &c.—Gan.

[2] *Adi* and *mukha*, *vadana*, *vaktra*, and other synonyms of face—the initial quantity of the progression, the first term: (that, from which as an origin the sequence commences.—Súr.)

Chaya, *prachaya* or *uttara*—the more (*adhika*—Súr.) or augment (*vriddhi*—Gang.) by which each term increases, the common difference. *Antya*, the last term. *Madhya*, the middle term. *Pada* or *gachchha*, the period, the number of terms: (so many days as the sequence reaches.—Súr.) *Sarva-dhana*, *sredhi-phala* or *ganita*—the amount of the whole, the sum of the progression. 'It is called *ganita*, because it is found by computation (*gananá*).—Gan.

day. Say quickly how many were given by him in half a month.

Statement: initial quantity 4; com. diff. 5; period 15.

Here, first term 4. Middle term 39. Last term 74. Sum 585.

121. Another example.[1] The initial term being seven, the increase five, and the period eight, tell me what the magnitudes of the middle and last terms are, and what the total sum is.

Statement: first term 7; com. diff. 5; period 8.

Answer: mean amount $\frac{49}{2}$. Last term 42. Sum 196.

Here, the period consisting of an even number of days, there is no middle day; wherefore half the sum of the days preceding and following the mean place, must be taken for the mean amount: and the rule is thus proved.

[See note to § 119.]

122. Rule[2]: half a stanza. The sum of the progression being divided by the period, and half the common difference multiplied by one less than the number of terms, being subtracted, the remainder is the initial quantity.[3]

[$s=\frac{n}{2}\left\{2a+(n-1)b\right\}=n\left\{a+\frac{n-1}{2}b\right\}; \therefore \frac{s}{n}-\frac{n-1}{2}b=a$, whence the rule.]

123. Example. We know the sum of the progression, one hundred and five; the number of terms, seven; the increase, three; tell us, dear boy, the initial quantity.

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[1] To exhibit an instance of an even number of terms, where there can consequently be no middle term (but a mean amount).—Gan.

[2] The difference, period and sum being given, to find the first term.—Gan. and Súr.

[3] The rule is the converse of the preceding.—Gan. and Súr.

Statement : com. diff. 3 ; period 7 ; sum 105.

Answer : first term, 6.

Rule[1]; half a stanza.[2] The sum being divided by the period, and the first term subtracted from the quotient, the remainder, divided by half of one less than the number of terms, will be the common difference.[3]

[$s = n\left\{a + \frac{n-1}{2}b\right\}$; $\therefore \left\{\frac{s}{n} - a\right\} \div \frac{n-1}{2} = b$, whence the rule.]

124. Example. On an expedition to seize his enemy's elephants, a king marched two *yojanas* the first day. Say, intelligent calculator, with what increasing rate of daily march he proceeded, he reaching his foe's city, a distance of eighty *yojanas*, in a week.

Statement : first term 2 ; period 7 ; sum 80.

Answer : com. diff. $\frac{22}{7}$.

125. Rule.[4] From the sum of the progression multiplied by twice the common increase, and added to the square of the difference between the first term and half that increase, the square root being extracted, this root less the first term and added to the (above-mentioned) portion of the increase, being divided by the increase, is pronounced[5] to be the period.

[The translation is rather obscure. A clearer rendering would be as follows :—" The sum of the progression multiplied by twice the common increase, being added to the square of the

[1] The first term, period and sum being known, to find the common difference which is unknown.—Gan.

[2] Second half of one, the first half of which contained the preceding rule, § 122.

[3] This rule also is converse of the foregoing.—Gan.

[4] The first term, common difference and sum being known, to find the period which is unknown.—Gan.

[5] By Brahmagupta and the rest.—Gan.

The rules are substantially the same ; the square being completed for the solution of the quadratic equation in the manner taught by Srídhara (cited in *Vija-ganita*, § 131) and by Brahmagupta.

difference between the first term and half that increase, the square root of the result is extracted ; this root less, &c."

We have $s=\frac{n}{2}\left\{2a+(n-1)b\right\}$;

whence $n=\frac{b-2a\pm\sqrt{\{(2a-b)^2+8sb\}}}{2b}$

(Todhunter's Algebra, Art. 454)

$=\frac{1}{b}\left\{\frac{b}{2}-a\pm\sqrt{\left\{(a-\frac{b}{2})^2+2sb\right\}}\right\}$, whence the rule, the *upper sign* only being taken by the author. He does not discuss the meaning of the *two* values of n.]

126. Example. A person gave three *drammas* on the first day, and continued to distribute alms increasing by two (a day) ; and he thus bestowed on the priests three hundred and sixty *drammas* : say quickly in how many days.

Statement : first term 3 ; com. diff. 2 ; sum 360.

Answer : period 18.

SECTION II.

GEOMETRICAL PROGRESSION.

127. Rule[1]: a couplet and a half. The period being an uneven number, subtract one, and note 'multiplicator'; being an even one, halve it, and note 'square,' until the period be exhausted. Then the produce arising from multiplication and squaring (of the common multiplier) in the inverse order from the last,[2] being lessened by one, the remainder divided by the common

[1] To find the sum of a progression, the increase being a multiplier.—Gan. In other words, to find the sum of an increasing geometrical progression.

[2] The last note is of course 'multiplicator.' For in exhausting the number of the period (when odd) you arrive at last at unity, an uneven number. The proposed multiplier (the common multiplicator of the progression) is therefore put in the last place; and the operations of squaring and multiplying by it are continued in the inverse order of the line of the notes.—Gan.

multiplier less one, and multiplied by the initial quantity, will be the sum of a progression increasing by a common multiplier.[1]

[Let a denote the first term and r the common ratio. Then, $a+ar+\ldots\ldots\ldots+ar^{n-1}=\frac{a(r^n-1)}{r-1}$. The first part of the rule is clumsily and obscurely stated. It is difficult to make out what the author means. He wants us to find r^n. Now if n is even and therefore of the form $2m$, we can find r^n by first squaring r, then squaring the square, and so on m times. If n is odd and therefore of the form $2m+1$, we can find r^n by first finding r^{2m} as before, and then multiplying the result by r. This is probably what the author means by the words multiplying and squaring. See the explanation of Ganesa in the footnotes.]

128. Example. A person gave a mendicant a couple of cowry shells first, and promised a twofold increase of the alms daily. How many *nishkas* did he give in a month ?

Statement : first term, 2 ; increasing multiplier, 2 ; period, 30.

Answer : 2147483646 cowries ; or 104857 *nishkas*, 9 *drammas*, 9 *panas*, 2 *kákinís*, and 6 shells.

129. Example. The initial quantity being two, my friend ; the daily augmentation, a threefold increase ; and the period, seven ; say what the sum in this case is.

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[1] The effect of squaring and multiplying, as directed, is the same as the continued multiplication of the multiplier for as many times as the number of the period. For dividing by the multiplier the product of the multiplication continued to the uneven number, equals the product of multiplication continued to one less than the number; and the extraction of the square root of a product of multiplication continued to the even number, equals continued multiplication to half that number. Conversely, squaring and multiplying equals multiplication for double and for one more time.—Gan.

Statement : first term, 2 ; increasing multiplier, 3 ; period, 7.

Answer : sum, 2186.

130—131. Rule :[1] a couplet and a half. The number of syllables in a verse being taken for the period, and the increase twofold, the produce of multiplication and squaring (as above directed, § 127) will be the number (of variations) of like verses.[2] Its square, and square's square, less their respective roots, will be (the variations) of alternately similar and of dissimilar verses (in tetrastics).[3]

[The rule refers specially to the example in § 132. It is a statement in a concrete shape of the following proposition :— The total number of combinations of n things taken, 1, 2, ... n at a time $= 2^n - 1$. (See note to § 113).]

132. Example. Tell me directly the number (of

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[1] Incidentally introduced in this place, showing a computation serviceable in prosody.—Súr. and *Mano*. To calculate the variations of verse, which are also found by the sum of permutations (§113).—Gan.

[2] Sanskrit prosody distinguishes metre in which the four verses of the stanza are alike, or the alternate ones only so, or all four dissimilar. Asiat. Res., Vol. X, syn. tab., v, vi and vii.

[3] The number of possible varieties of verse found by the rule of permutation (§113) is the same as the continued multiplication of two : this number being taken, because the varieties of syllables are so many, long and short. Accordingly this is assumed for the common multiplier. The product of its continued multiplication is to be found by this method of squaring and multiplying (§127) ; assuming for the period a number equal to that of syllables in the verse The varieties of similar verses are the same as those of one verse containing twice as many syllables ; and the changes in the four verses are the same as those of one verse comprising four times as many syllables, excepting, however, that these permutations embracing all the possible varieties, comprehend those of like and half-alike metre. Wherefore the number first found is squared, and this again squared, corresponding to twice or four times the number of places ; and the roots of these squares are subtracted [for obtaining the varieties of alternately like and dissimilar verses respectively,—Ed.]—Gan.

varieties) of like, alternately like, and dissimilar verses respectively, in the metre named *anushṭubh*.[1]

Statement : increasing multiplier, 2 ; period, 8.

Answer : variations of like verses, 256 ; of alternately alike verses, 65280 ; of dissimilar verses, 4294901760.

[The total number of syllables in the four *charanas* of the *anushtubh* metre being 32 (8 to each *charana*), the possible varieties of arrangements of long and short syllables in the metre are $2^{32} = 4294967296$. (See note to § 113). These evidently include cases of (*a*) all like, and (*b*) alternately like *charanas*. To find the number of these cases, we find the number of varieties of the syllables in *two charanas* ; which number is 2^{16} or 65536. It is clear that if we place each one of these varieties under itself, we shall get all the cases included in (*a*) and (*b*). Hence the total number of cases in (*a*) and (*b*) is 65536. Of these the number of cases in (*a*) clearly is the number of varieties that may occur in one *charana* $= 2^8 = 256$. Consequently the number of cases in (*b*) is $65536 - 256$ or 65280. Subtracting 65536 from 4294967296 we get 4294901760, which is considered as the number of *dissimilar* verses or *charanas*. It is to be observed, however, that these last include cases in which the first two *charanas* are like, as also the last two ; or cases in which the first two are like, but the last two unlike ; and so forth. These cases are not separately considered.]

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[1] Asiat. Res., Vol. X, p. 438 ; syn. tab., p. 469.

CHAPTER VI.

PLANE FIGURE.[1]

133. Rule. A side[2] is assumed. The other side in

[1] *Kshetra-vyavahára*, determination of plane figure. *Kshetra*, as expounded by Ganesa, signifies plane surface bounded by lines, straight or curved; as triangle, &c. *Vyavahára* is the ascertainment of its dimensions, as diagonal, perpendicular, area, &c. Ganesa says plane figure is fourfold; triangle, quadrangle, circle and bow. Triangle (*tryasra*, *trikona* or *tribhuja*) is a figure containing (*tri*) three (*asra* or *kona*) angles, and consisting of as many (*bhuja*) sides. Quadrangle or tetragon (*chaturasra*, *chatushkona*, *chaturbhuja*), is a figure comprising (*chatur*) four (*asra*, &c.) angles or sides. The circle and bow, he observes, need no definition. Triangle is either (*játya*) right-angled, as that which is first treated of in the text; or it is (*tribhuja*) trilateral (and oblique) like the fruit of the *sringáta* (Trapa natans). This again is distinguished according as the (*lamba*) perpendicular falls within or without the figure: *viz.*, *antarlamba*, acute-angled; *bahirlamba*, obtuse-angled. Quadrangle also is in the first place twofold: with equal, or with unequal, diagonals. [This is not a proper classification.—Ed.] The first of these, or equidiagonal tetragon (*sama-karna*), comprises four distinctions: 1st, *sama-chaturbhuja*, equilateral, a square; 2d. *vishama-chaturbhuja*, a trapezium; 3d. *áyata-dirgha-chaturasra*, an oblique parallelogram; [this is not correct; for a parallelogram with equal diagonals must be either a rectangle or a square, so that this 3d. cannot be a distinct species.—Ed.]; 4th. *áyata-samalamba*, oblong with equal perpendiculars, *i.e.*, a rectangle. The second sort of quadrangle, or the tetragon with unequal diagonals (*vishamakarna*), embraces six sorts: 1st. *sama-chaturbhuja*, equilateral, a rhombus; 2d. *sama-tribhuja*, having three equal sides; 3d. *sama-dwi-dwi-bhuja*, consisting of two pairs of equal sides, a rhomboid; 4th *sama-dwi-bhuja*, having two equal sides; 5th. *vishama-chaturbhuja*, composed of four unequal sides, a trapezium; 6th. *sama-lamba*, having equal perpendiculars, a trapezoid. The several sorts of figures, observes the commentator, are fourteen; the circle and bow being but of one kind each. He adds, that pentagons (*pancházra*), &c. comprise triangles (and are reducible to them).

[2] *Báhu*, *dosh*, *bhuja*, and other synonyms of arm are used for the leg of a triangle, or side of a quadrangle or polygon; so called, as resembling the human arm.—Gan. and Súr.

the rival direction is called the upright,[1] whether in a triangle or tetragon, by persons conversant with the subject.

134. The square root of the sum of the squares of those legs is the diagonal.[2] The square root, extracted from the difference of the squares of the diagonal and side, is the upright ; and that, extracted from the difference of the squares of the diagonal and upright, is the side.[3]

[Euclid I. 47.]

135.[4] Twice the product of two quantities, added to the square of their difference, will be the sum of their squares. The product of their sum and difference will be the difference of their squares : as must be everywhere understood by the intelligent calculator.[5]

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[1] Either leg being selected to retain this appellation, the others are distinguished by different names. That which proceeds in the opposite direction, meaning at right angles, is called *koti*, *uchchhráya*, *uchchhriti*, or any other term signifying upright or elevated. Both are alike sides of the triangle or of the tetragon, differing only in assumed situation and name.—Gan. and Súr.

[2] A thread or oblique line from the two extremities of the legs, joining them, is the *karna*, also termed *sruti*, *sravana*, on any other word signifying ear. It is the diagonal of a tetragon.—Súr., Rang., &c. Or, in the case of a triangle, it is the diagonal of the parallelogram, whereof the triangle is the half : and is the hypotenuse of a right-angled triangle.

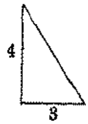
[3] The rule concerns (*játya*) right-angled triangles. The proof is given both algebraically and geometrically by Ganesa (*upapatti avyakta-kriyayá*, proof by algebra ; *kshetragatopapatti*, geometrical demonstration) ; and the algebraical proof is also given by Súryadása. Ranganátha cites one of those demonstrations from his brother Kamalákara, and the other from his father Nrisinha, in the *Vártika*, or critical remarks on the (*Vásaná*) annotations of the *Siromani* ; and censures the *Sringára-tilaka* for denying any proof of the rule besides experience. Bháskara has himself given a demonstration of the rule in his *Víja-ganita*, § 146.

[4] A stanza of six *charanas* of *anushtubh* metre.

[5] Ganesa here also gives both an algebraic and a geometrical proof of the latter rule ; and an algebraical one only of the first. See *Víja-ganita* under § 148, whence the latter demonstration is borrowed ; and § 147, where the first of the rules is given and demonstrated.

[$2ab + (a-b)^2 = a^2 + b^2$.
$(a+b)\ (a-b) = a^2 - b^2$.
Geometrical proofs of these formulæ are furnished by Euc. II. 5 and 9. The object of introducing them here is to facilitate the calculations required in § 134.]

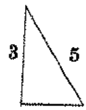
136. Example. Where the upright is four and the side three, what is the hypotenuse? Tell me also the upright from the hypotenuse and side; and the side from the upright and hypotenuse.



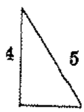
Statement : side 3 ; upright 4.

Sum of their squares 25. Or, the product of the sides, doubled, 24 ; square of the difference, 1 ; added together, 25. The square root of this is the hypotenuse 5.

Difference of the squares 5 and 3 is 16. Or the sum 8 multiplied by the difference 2, makes 16. Its square root is the upright 4.

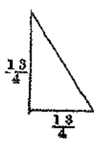


Difference of squares, found as before, 9. Its square root is the side 3.



137. Example. Where the side measures three and a quarter, and the upright, as much; tell me quickly, mathematician, what the length of the hypotenuse is.

Statement : side $\frac{13}{4}$; upright $\frac{13}{4}$. Sum of the squares $\frac{338}{16}$ or $\frac{169}{8}$. Since this has no (assignable) root, the hypotenuse is a surd. A method of finding its approximate root follows :—



138. Rule. From the product of numerator and

denominator,[1] multiplied by any large square number assumed, extract the square root: that, divided by the denominator taken into the root of the multiplier, will be an approximation.

The square of the above hypotenuse, $\frac{169}{8}$ (is proposed). The product of its numerator and denominator is 1352. Multiplied by a myriad (the square of a hundred) the product is 13520000. Its root is 3677 nearly.[2] This divided by the denominator taken into the square root of the multiplier, *viz.*, 800, gives the approximate root $4\frac{477}{800}$. It is the hypotenuse. So in every similar instance.

[$\sqrt{\frac{a}{b}} = \frac{\sqrt{a \times b}}{b} = \frac{\sqrt{a \times b \times c^2}}{bc}$.

The object of multiplying the product of numerator and denominator by a large square number (being some power of ten, as the above process shows), and then taking the square root approximately is practically to get the square root to a certain number of decimal places. Bháskara, however, does not use the decimal notation which was probably not known in his time, and expresses the result as a fraction. In the above example, the result obtained will be found to be correct to two decimal places.]

139. Rule.[3] A side is put. From that multiplied by twice some assumed number, and divided by one less than the square of the assumed number, an upright is obtained. This, being set apart, is multiplied by the arbitrary number, and the side as put is subtracted;

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[1] If the surd be not a fraction, unity may be put for the denominator, and the rule holds good.—Gan.

[2] The remainder being unnoticed.

[3] Either the side or upright being given, to find the other two sides.—Súr. To find the upright and hypotenuse from the side; or the side and hypotenuse from the upright.—Gan. The problem is an indeterminate one, as is intimated by the author.

the remainder will be the hypotenuse. Such a triangle is termed right-angled.[1]

[Let a denote the given side, and n the assumed number. Then proceeding by the rule, we get $\frac{2an}{n^2-1}$ for upright, and $\frac{2an}{n^2-1} \times n - a = a\frac{n^2+1}{n^2-1}$ for hypotenuse. To verify this we have $a^2 + \left(\frac{2an}{n^2-1}\right)^2 = \frac{a^2}{(n^2-1)^2}\left\{(n^2-1)^2 + 4n^2\right\} = \frac{a^2(n^2+1)^2}{(n^2-1)^2}$ $= \left(a\frac{n^2+1}{n^2-1}\right)^2$. The proof of this rule given by Súryadása shows how these expressions for the upright and hypotenuse are arrived at, although it is rather difficult to follow it. The quantities $2n$, n^2-1, n^2+1 may be taken to represent the upright, side and hypotenuse of a right-angled triangle, because $(2n)^2 + (n^2-1)^2 = (n^2+1)^2$. Now consider another right-angled triangle similar to the above, the side being a. Then, since the sides of the two triangles are proportional, the upright of the second triangle will obviously be $\frac{2an}{n^2-1}$. Again, as $n \times$ upright of the first triangle = its side + its hypotenuse, so $n \times$ upright of the second triangle = its side + its hypotenuse. Thus, $n \times \frac{2an}{n^2-1} = a +$ hypotenuse, whence hypotenuse $= n \times \frac{2an}{n^2-1} - a$. Thus we see how the expressions are got.]

140. Or a side is put. Its square, divided by an arbitrary number, is set down in two places: and the arbitrary number being added and subtracted, and the sum and difference halved, the results are the hypotenuse and upright.[2] Or, in like manner, the side and

[[1] Colebrooke uses the word *rectangular*. But the more usual word in modern geometry is *right-angled*.—Ed.]

[2] Assume any number for the difference between the upright and hypotenuse. The difference of their squares (which is equal to the square of the given side) being divided by that assumed difference, the quotient is the sum of the upright and hypotenuse. For the difference of the squares is

hypotenuse may be deduced from the upright. Both results are rational quantities.

[Let a denote the given side, and n the assumed number. Then by the rule we have $\frac{1}{2}\left(\frac{a^2}{n}+n\right)$ for hypotenuse, and $\frac{1}{2}\left(\frac{a^2}{n}-n\right)$ for upright. To verify this we have $a^2+\frac{1}{4}\,\frac{(a^2-n^2)^2}{n^2}$ $=\frac{4a^2n^2+(a^2-n^2)^2}{4n^2}=\frac{(a^2+n^2)^2}{4n^2}=\left\{\frac{1}{2}\left(\frac{a^2}{n}+n\right)\right\}^2$. Ganesa gives an elegant demonstration of this rule, based on the fact that the assumed number n is the difference between the hypotenuse and upright, as is obviously the case. See foot-note.]

141. Example. The side being in both cases twelve, tell quickly by both methods, several uprights and hypotenuses, which shall be rational numbers.

Statement: side 12; assumption 2. The side, multiplied by twice that, *viz.*, 4, is 48. Divide by the square of the arbitrary number less one, *viz.*, 3, the quotient is the upright 16. This upright multiplied by the assumed number is 32, from which subtract the given side; the remainder is the hypotenuse 20.

Assume 3. The upright is 9, and the hypotenuse 15. Or, putting 5, the upright is 5, and the hypotenuse 13.

By the second method: the side, as put, 12. Its square 144. Divide by 2, the arbitrary number being 2, the quotient is 72. Add and subtract the arbitrary number, and halve the sum and difference. The hypotenuse and upright are found, *viz.*, hypotenuse 37, upright 35.

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equal to the product of the sum and difference of the roots (§ 135). The upright and hypotenuse are therefore found by the rule of concurrence (§ 55).—Gan.

Assume 4. The upright is 16, and the hypotenuse 20. Assuming 6, the upright is 9, and the hypotenuse 15.[1]

142. Rule.[2] Twice the hypotenuse taken into an arbitrary number, being divided by the square of the arbitrary number added to one, the quotient is the upright. This taken apart is to be multiplied by the number put: the difference between the product and the hypotenuse is the side.

[Let a denote the hypotenuse, and n the assumed number. Then, by the rule, the upright is $\frac{2an}{n^2+1}$, and the side, $\frac{2an^2}{n^2+1}-a$ $=a\frac{n^2-1}{n^2+1}$. For $\left(\frac{2an}{n^2+1}\right)^2+\left(a\frac{n^2-1}{n^2+1}\right)^2=a^2$. The proof of this rule given by Súryadása is exactly similar to that of the rule in §139. See note to §139.]

143. Example. The hypotenuse being measured by eighty-five, say promptly, learned man, what uprights and sides will be rational.

Statement: hypotenuse 85. This doubled is 170, and multiplied by an arbitrary number two is 340. This, divided by the square of the arbitrary number added to one, *viz.*, 5, is the upright 68. This upright multiplied by the arbitrary number makes 136; and subtracting the hypotenuse, the side comes out 51. Or putting four, the upright will be 40, and the side 75.

144. Rule. Or else the hypotenuse is doubled and divided by the square of an assumed number added to one. The hypotenuse less that quotient is the upright.

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[1] In like manner, if the upright be given 16, its square 256 divided by the arbitrary number 2 is 128. The arbitrary number, subtracted and added, makes 126 and 130; which halved give the side 63, and the hypotenuse 65.—Gang. and Súr.

[2] From the hypotenuse given, to find the side and upright in rational numbers.—Gan. The problem is an indeterminate one.

The same quotient multiplied by the assumed number is the side.[1]

The same hypotenuse 85. Putting two, the upright and side are 51 and 68. Or, with four, they are 75 and 40.

Here the distinction between side and upright is in name only, and not essential.

[Taking a and n as in § 142, we get $a-\frac{2a}{n^2+1}=a\frac{n^2-1}{n^2+1}$ for upright, and $\frac{2an}{n^2+1}$ for side. Verification as in § 142. To see how these expressions are arrived at, take $2n$, n^2-1 and n^2+1 for the side, upright and hypotenuse of a right-angled triangle, and proceed as in § 139, note.]

145. Rule.[2] Let twice the product of two assumed numbers be the upright; and the difference of their squares, the side: the sum of their squares will be the hypotenuse, and a rational number.

[Let a and b be the assumed numbers.

Then $2ab$ is the upright, and a^2-b^2 the side. The hypotenuse is $\sqrt{(2ab)^2+(a^2-b^2)^2}=a^2+b^2$.

Thus the three sides are all rational. Ganesa gives a proof of this rule after the manner of the *Vija-ganita;* but it is very obscure and cannot be easily followed.]

146. Example. Tell quickly, friend, three numbers, none being given, with which as upright, side and hypotenuse, a right-angled triangle may be (constructed.)

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[1] This and the preceding rule are founded on the same principle, differing only in the order of the operation and names of the sides: the same numbers come out for the side and upright in one mode, which were found for the upright and side by the other.

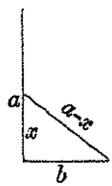
[2] Having taught the mode of finding a third side from any two of hypotenuse, upright and side; and in like manner from one, the other two; the author now shows a method of finding all three rational (none being given). —Gan. The problem is an indeterminate one.

Let two numbers be put, 1 and 2. From these, the side, upright and hypotenuse are found, 4, 3, 5. Or, putting 2 and 3, the side, upright and hypotenuse deduced from them are, 12, 5, 13. Or let the assumed numbers be 2 and 4 : from which will result 16, 12, 20. In like manner, manifold (answers are obtained).

147. Rule.[1] The square of the ground intercepted between the root and tip is divided by the (length of the) bambu, and the quotient severally added to, and subtracted from, the bambu : the moieties (of the sum and difference) will be the two portions of it representing hypotenuse and upright.[2]

[The rule bears reference to the example which follows.

Let a denote the height of the bambu, b, the distance between root and tip, and x the height at which the bambu is broken.



Then, $b^2 = (a-x)^2 - x^2$

$$\therefore \frac{b^2}{(a-x)+x} = (a-x)-x$$

$$i.e.\ \frac{b^2}{a} = (a-x)-x$$

also $a = (a-x)+x$, identically.

Hence $\frac{1}{2}\left(a+\frac{b^2}{a}\right) = a-x$ or hypotenuse, and $\frac{1}{2}\left(a-\frac{b^2}{a}\right) = x$ or upright, whence the rule. The above proof is the same as that given by Ganesa. See foot-note.]

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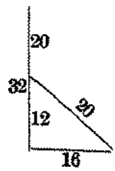
[1] The sum of hypotenuse and upright being known, as also the side, to discriminate the hypotenuse and upright.—Gan.

[2] The height from the root to the fracture is the upright. The remaining portion of the bambu is the hypotenuse. The whole bambu, therefore, is the sum of hypotenuse and upright. The ground intercepted between the root and tip is the side: it is equal to the square root of the difference between the squares of the hypotenuse and upright. Hence the square of the side, divided by the sum of hypotenuse and upright, is their difference (§ 135). With these (sum and difference) the upright and hypotenuse are found by the rule of concurrence (§ 55) —Gan.

148. Example. If a bambu, measuring thirty-two cubits and standing upon level ground, be broken in one place by the force of the wind, and the tip of it meet the ground at sixteen cubits : say, mathematician, at how many cubits from the root it is broken.

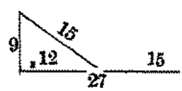
Statement. Bambu 32. Interval between the root and tip of the bambu, 16. It is the side of the triangle. Proceeding as directed, the upper and lower portions of the bambu are found to be 20 and 12.



149. Rule.[1] The square (of the height) of the pillar is divided by the distance of the snake from his hole; the quotient is to be subtracted from that distance. The meeting of the snake and peacock is from the snake's hole half the remainder, in cubits.

150. Example. A snake's hole is at the foot of a pillar, nine cubits high, and a peacock is perched on its summit. Seeing a snake at the distance of thrice the pillar gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole they meet, both proceeding an equal distance.

Statement. Pillar 9. It is the upright. Distance of the snake from his hole, 27. It is the sum of hypotenuse and side. Proceeding



[1] The sum of the side and hypotenuse being known, as also the upright, to discriminate the hypotenuse and side.—Gan. The rule bears reference to the example which follows. The principle is the same as that of the preceding rule.

as directed, the distance between the hole and the place of meeting is found to be 12 cubits.[1]

[The principle of the rule in § 149 is, as Colebrooke observes, and as is also evident from the example in § 150, the same as that of the rule in § 147. The peacock is supposed not to change his direction, and to pounce in such a direction that the distance traversed by him being the hypotenuse of a right-angled triangle, is equal to the distance traversed by the snake. Practically, however, such a thing does not happen; but the bird of prey changes its direction at every instant, and describes a curved path known as the curve of pursuit. See Tait and Steele's Dynamics of a Particle, Art. 33.

Let a denote the distance of the snake from the hole, b the height of the pillar, and x the distance required.

Then, $b^2=(a-x)^2-x^2$, whence as in § 147, $x=\frac{1}{2}\left(a-\frac{b^2}{a}\right)$. Hence the rule.]

151. Rule.[2] The quotient of the square of the side divided by the difference between the hypotenuse and upright is twice set down; and the difference is subtracted from the quotient (in one place) and added to it (in the other). The moieties (of the remainder and sum) are in their order the upright and hypotenuse.[3]

This[4] is to be generally applied by the intelligent mathematician.

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[1] Subtracted from the sum of hypotenuse and side, this leaves 15 for the hypotenuse. The snake had proceeded the same distance of 15 cubits towards his hole, as the peacock in pouncing upon him. Their progress is therefore equal.—Súr.

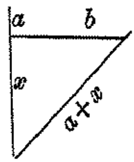
[2] The difference between the hypotenuse and upright being known, as also the side, to find the upright and hypotenuse.—Gan.

[3] The demonstration, distinctly set forth under a preceding rule, is applicable to this.—Gan.

[4] Beginning from the instance of the broken bambu (§ 147) and including what follows.—Gan.

[The demonstration of the rule in §147 applies to this rule as Ganesa observes.

Let a denote the difference between hypotenuse and upright, b the side, and x the upright.

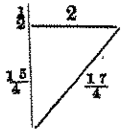


Then, $b^2=(a+x)^2-x^2$, whence as in § 147, $x=\frac{1}{2}\left(\frac{b^2}{a}-a\right)$, and $a+x=\frac{1}{2}\left(\frac{b^2}{a}+a\right)$. Hence the rule.]

152. Friend, the space between the lotus (as it stood) and the spot where it submerged, is the side. The lotus as seen (above water) is the difference between the hypotenuse and upright. The stalk is the upright, for the depth of water is measured by it. Say, what the depth of the water is.[1]

153. Example.[2] In a certain lake swarming with ruddy geese[3] and cranes, the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind it gradually advanced, and was submerged at the distance of two cubits. Compute quickly, mathematician, the depth of the water.

Statement. Diff. of hypotenuse and upright, $\frac{1}{2}$ cubit. Side 2 cubits. Proceeding as directed, the upright is found $\frac{15}{4}$. It is the depth of the water. Adding to it the height of the bud, the hypotenuse comes out $\frac{17}{4}$.



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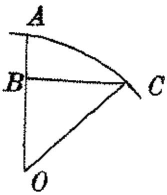
[1] The sides constituting the figure in the example which follows, are here set forth, to assist the apprehension of the student.—Súr. and Gan.

[2] [This example is inserted in Barnard Smith's Arithmetic, Appendix, p. 300.—Ed.]

[3] Anas Casarca.

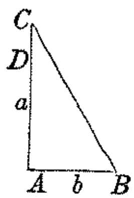
[Let O be the root of the lotus, A its tip, and C the point on the surface of the water where it is submerged. Then, while it advances by the force of the wind, O remains fixed, and the lotus describes an arc of a circle, of which O is the centre, and OA the radius. Hence $OC=OA$. Then, as the author himself explains in § 152, the solution will follow from the method of § 151, where we have only to substitute $\frac{1}{2}$ for a, and 2 for b.]



154. Rule.[1] The height of the tree multiplied by its distance from the pond, is divided by twice the height of the tree added to the space between the tree and the pond : the quotient will be the measure of the leap.

[The rule refers to the example which follows.

Let D be the top of the tree, and B the position of the pond. The first ape is supposed to descend from D to A, and then to go from A to B; while the second ape is supposed to jump vertically upwards from D to C, and then to leap directly from C to B. Now let $AD=a$, $AB=b$, and $CD=x$, which is required. Then by the question, we have $x+\sqrt{(a+x)^2+b^2}=a+b$;



$$\therefore\ (a+x)^2+b^2=(a+b)^2-2\ (a+b)\ x+x^2,$$

whence $\qquad x=\dfrac{ab}{2a+b}$. Hence the rule.]

155. Example. From a tree a hundred cubits high, an ape descended and went to a pond two hundred cubits distant : while another ape, vaulting to some height off the tree, proceeded with velocity diagonally

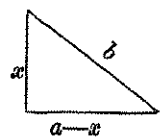
[1] The sum of the hypotenuse and upper portion of the upright being given, and the lower portion being known, as also the side : to discriminate the upper portion of the upright from the hypotenuse.—Gan. As in several preceding instances, a reference to the example is requisite to the understanding of the rule.

to the same spot. If the space travelled by them be equal, tell me quickly, learned man, the height of the leap, if thou have diligently studied calculation.

Statement. Tree 100 cubits. Distance of it from the pond 200. Proceeding as directed, the height of the leap comes out 50.

156. Rule.[1] From twice the square of the hypotenuse subtract the sum of the upright and side multiplied by itself, and extract the square root of the remainder. Set down the sum twice, and let the root be subtracted in one place and added in the other. The moieties will be measures of the side and upright.[2]

[Let a denote the sum of side and upright, and b the hypotenuse. Also let x denote the upright. Then we have evidently, $(a-x)^2+x^2=b^2$;



$\therefore\ 2x^2-2ax+a^2-b^2=0,$

whence $x=\dfrac{a\pm\sqrt{2b^2-a^2}}{2}$.

If we take the upper sign, then $a-x=\dfrac{a-\sqrt{2b^2-a^2}}{2}$, and

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[1] Hypotenuse being known, as also the sum of the side and upright, or their difference; to discriminate those sides.—Gan.

[2] In like manner, the difference of the side and upright being given, the same rule is applicable.—Gan. [A slight variation will be necessary; see note to § 158.—Ed.] The principle of the rule is this: the square of the hypotenuse is the sum of the squares of the sides. But the sum of the squares with twice the product of the sides added to it is the square of the sum; and with the same subtracted is the square of the difference. Hence cancelling equal quantities affirmative and negative, twice the square of the hypotenuse will be the sum of the squares of the sum and difference. Therefore, subtracting from twice the square of hypotenuse the square of the sum, the remainder is the square of the difference; or conversely, subtracting the square of the difference, the residue is the square of the sum. The square root is the sum or difference. With these, the sides are found by the rule of concurrence.—Gan. and Súr.

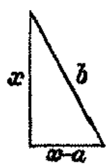
if we take the lower sign, then $a-x=\frac{a+\sqrt{2b^2-a^2}}{2}$. It is evident that we may take either sign. The reason for the rule is obvious. The method of solving an adfected quadratic equation by completing the square has been mentioned before. See §§ 62–63. Interesting geometrical interpretations of the expressions for x and $a-x$ are given by Ganesa and Súryadása. See foot-note.]

157. Example. Where the hypotenuse is seven above ten; and the sum of the side and upright, three above twenty; tell them to me, my friend.

Statement: Hypotenuse 17; sum of side and upright 23. Proceeding as directed, the side and upright are found 8 and 15.

158. Example.[1] Where the difference of the side and upright is seven, and hypotenuse is thirteen, say quickly, eminent mathematician, what the side and upright are.

Statement: hypotenuse 13; difference of side and upright 7. Proceeding as directed, the side and upright come out 5 and 12.



[Let x denote the upright, and a the diff. between upright and side, the upright being supposed > side. Then $x-a$ will denote the side; and we evidently have $x^2+(x-a)^2=b^2$, whence as in § 156, $x=\frac{a\pm\sqrt{2b^2-a^2}}{2}$. But here we must take the upper sign alone, since $x-a$ is necessarily positive. Thus we have $x=\frac{\sqrt{2b^2-a^2}+a}{2}$, and $x-a=\frac{\sqrt{2b^2-a^2}-a}{2}$.

[1] This example of a case where the difference of the sides is given, is omitted by Súryadása, but noticed by Ganesa. Copies of the text vary: some containing, and others omitting, the instance.

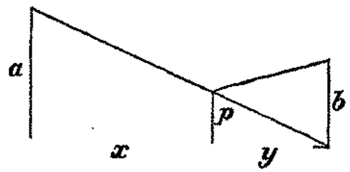
From these values it is clear that the rule in § 156 must be slightly varied in order to be applicable to the present case.]

159. Rule.[1] The product of two erect bambus being divided by their sum, the quotient is the perpendicular[2] from the junction (intersection) of threads passing reciprocally from the root (of one) to the tip (of the other). The two bambus, multiplied by an assumed base, and divided by their sum, are the portions of the base on the respective sides of the perpendicular.

[From similar triangles (see figure) we have

$$\frac{p}{a}=\frac{y}{x+y},$$

$$\frac{p}{b}=\frac{x}{x+y};$$



$$\therefore\ p\left(\frac{1}{a}+\frac{1}{b}\right)=1, \text{ and } \therefore\ p=\frac{ab}{a+b}.$$

Thus p is independent of x and y, provided a and b be given. Again, let $x+y=k$, any assumed number.

Then $x=\frac{pk}{b}=\frac{ak}{a+b}$,

and $y=\frac{pk}{a}=\frac{bk}{a+b}$.

This rule shows that the property of similar triangles was known. See Bháskara's remark at the end of § 160, and the proof given by Ganesa, cited in the foot-note, which is slightly different and more cumbrous.]

160. Example. Tell the perpendicular drawn from the intersection of strings stretched mutually from the roots to the summits of two bambus, fifteen and ten cubits high, standing upon ground of unknown extent.

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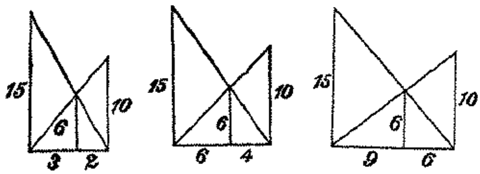
[1] Having taught fully the method of finding the sides in a right-angled triangle, the author next propounds a special problem.—Gan. To find the perpendicular, the base being unknown.—Súr.

[2] *Lamba, avalamba, valamba, adholamba*, the perpendicular.

Statement : bambus 15, 10. The perpendicular is found 6.

Next to find the segments of the base. Let the ground be assumed 5 ; the



segments come out 3 and 2. Or putting 10, they are 6 and 4. Or taking 15, they are 9 and 6. See the figures. In every instance the perpendicular is the same,[1] *viz.*, 6.

The proof is in every case by the rule of three : if with a side equal to the base, the bambu be the upright, then with the segment of the base, what will be the upright ?[2]

161. Aphorism.[3] That figure, though rectilinear, of which sides are proposed by some presumptuous

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[1] However the base may vary by assuming a greater or less quantity for it, the perpendicular will always be the same.—Gan.

[2] On each side of the perpendicular, are segments of the base relative to the greater and smaller bambus, and larger or less analogously to them. Hence this proportion : "If with the sum of the bambus, this sum of the segments equal to the entire base be obtained, then, with the smaller bambu, what is had ?" [This proportion cannot be at once obtained easily, but may be got by dividing corresponding members of the first two equations in the note to §159, whence we have $\frac{a}{b}=\frac{x}{y}$, and therefore $\frac{a+b}{x+y}=\frac{b}{y}$.—Ed.] The answer gives the segment which is relative to the least bambu. Again : "If with a side equal to the whole base, the higher bambu be the upright, then with a side equal to the segment found as above, what is had ?" The answer gives the perpendicular let fall from the intersection of the threads. Here a multiplicator and a divisor equal to the entire base are both cancelled as equal and contrary : and there remain the product of the two bambus for numerator and their sum for denominator. Hence the rule.—Gan.

[3] The aphorism explains the nature of impossible figures proposed by dunces.—Súr. It serves as a definition of plane figure (*kshetra*).—Gan. In a triangle or other plane rectilinear figure, one side is always less than the sum of the rest. If equal, the perpendicular is nought, and there is no complete figure. If greater, the sides do not meet.—Súr. Containing no area, it is no figure.—*Kaumudí* cited by Ranganátha.

person, wherein one side[1] exceeds or equals the sum of the other sides, may be known to be no figure.

[This follows from Euclid I. 20. Any side of a triangle or of any polygon must be less than the sum of the remaining sides. Hence the numbers 2, 3, 6, 12 cannot represent the sides of a quadrilateral.]

162. Example. Where sides are proposed two, three, six and twelve in a quadrilateral, or three, six and nine in a triangle, by some presumptuous dunce, know it to be no figure.

Statement. The figures are both incongruous. Let straight rods exactly of the lengths of the proposed sides be placed on the ground, the incongruity will be apparent.[2]

163—164. Rule[3] in two couplets. In a triangle, the sum of two sides being multiplied by their difference, is divided by the base[4]; the quotient is subtracted from, and added to, the base which is twice set down: and being halved, the results are segments corresponding to those sides.[5]

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[1] The principal or greatest side.—Gan. *Kaum.* Rang.

[2] The rods will not meet.—Súr.

[3] In any triangle to find the perpendicular, segments and area. This is introductory to a fuller consideration of areas.—Gan. and Súr.

[4] *Bhúmi, bhú, ku, mahí* or any other term signifying earth; the ground or base of a triangle or other plane figure. Any one of the sides is taken for the base, and the rest are termed simply sides. Ganesa restricts the term to the greatest side. See note § 168.

Lamba, &c., the perpendicular. See note § 159. *Abádhá, abadhá, avabádhá*, segment of the base made by the perpendicular. These are terms introduced by earlier writers. These segments are internal in an acute-angled triangle, but external in an obtuse-angled one. *Phala, ganita, kshetra-phala, samakoshtha-miti*: the measure of like compartments, or number of equal squares of the same denomination (as cubit, fathom, finger, &c.) in which the dimension of the side is given; the area or superficial content.—Gan. and Súr.

[5] The relative or corresponding segments. The smaller segment answers to the less side, and the larger to the greater side.—Gan.

The square root of the difference of the squares of the side and its own segment of the base becomes the perpendicular. Half the base multiplied by the perpendicular[1] is in a triangle the exact[2] area.[3]

[$x+y$ is given, as also a and b.

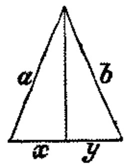
We have $a^2-b^2=x^2-y^2$,

$$\therefore \frac{a^2-b^2}{x+y}=x-y.$$

Then $(x+y)+(x-y)=2x$,

and $(x+y)-(x-y)=2y$;

and half of these results give x and y.



Also perpendicular $=\sqrt{a^2-x^2}$,

and area $=\frac{1}{2}\times$ base $\times$ altitude.

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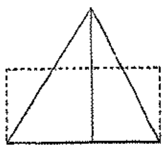
[1] Or half the perpendicular taken into the base.—Gan.

[2] *Sphuta-phala*, distinct or precise area; opposed to *asphuta*—or *sthúla-phala*, indistinct or gross area. See § 167.

[3] Demonstration. In both the right-angled triangles formed in the proposed triangle, one on each side of the perpendicular, this line is the upright; the side is hypotenuse, and the corresponding segment is side. Hence, subtracting the square of the perpendicular from the square of the side, the remainder is the square of the segment. So, subtracting the square of the other side, there remains the square of the segment answering to it. Their difference is the difference of the squares of the segments, and is equal to the difference of the squares of the sides, since an equal quantity has been taken from each; for any two quantities less an equal quantity have the same difference. It is equal to the product of the sum and difference of the simple quantities. Therefore, the sum of the sides multiplied by their difference is the difference of the squares of the segments. But the base is the sum of the segments. The difference of the squares, divided by that, is the difference of the segments. From which by the rule of concurrence (§ 55) the segments are found.

The square root of the difference between the squares of the side and segment (taken as hypotenuse and side) is the upright or perpendicular.

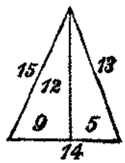
Dividing the triangle by a line across the middle (of the perpendicular), and placing the two parts of the upper portion disjoined by the perpendicular on the two sides of the lower portion (as in the annexed figure), an oblong is formed in which the half of the perpendicular is one side, and the base is the other. Wherefore half the perpendicular multiplied by the base is the area or number of equal compartments. Or, half the base multiplied by the



The above is practically the demonstration given by Ganesa, although it is rather long as it is expressed in words.]

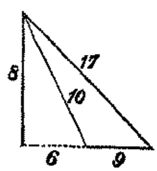
165. Example. In a triangular figure in which the base is fourteen and its sides thirteen and fifteen, tell quickly the length of the perpendicular, the segments, and the dimension by like compartments termed area.

Statement: base 14; sides 13 and 15. Proceeding as directed, the segments are found, 5 and 9; the perpendicular, 12; the area 84.



166. Example. In a triangle, wherein the sides measure ten and seventeen, and the base nine, tell me promptly, expert mathematician, the segments, perpendicular, and area.

Statement: base 9; sides 10 and 17. By the rule in §163, the quotient found is 21. This cannot be subtracted from the base; wherefore the base is subtracted from it. Half the remainder is the segment, 6, and is negative, that is to say, in the contrary direction.[1] (See figure.) Thus the two segments are found 6 and 15.



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perpendicular is just so much. In an obtuse-angled triangle also, the base multiplied by half the perpendicular is the area.—Gan.

[The proof given by Súryadása is practically the same as the above, and so we omit it here.—Ed.]

[1] When the perpendicular falls without the base, as overpassing the angle in consequence of the side exceeding the base, the quotient found by the rule in § 163 cannot be taken from the base; for both origins of sides are situated in the same quarter from the fall of the perpendicular. Therefore subtracting the base from the quotient, half the residue is the segment and situated on the contrary side. being negative. Wherefore, as both segments stand on the same side, the smaller is comprehended in the greater, and, in respect of it, is negative. Thus all is congruous and unexceptionable.—Gan.

From which, both ways too, the perpendicular comes out 8. The area is 36.

[As the commentators observe, the segments of the base made by the perpendicular in an obtuse-angled triangle are external, and their algebraical sum is their arithmetical difference.]

167. Rule.[1] Half the sum of all the sides is set down in four places; and the sides are severally subtracted. The remainders being multiplied together, the square root of the product is the area inexact in the quadrilateral, but pronounced exact in the triangle.[2]

[Let s denote the sermi-perimeter of a triangle. Then area $=\sqrt{s(s-a)(s-b)(s-c)}$. See Todhunter's Trigonometry, Art. 247. This rule does not apply to the case of a quadrilateral. In fact, a quadrilateral in general is not determined by the

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When the sum of the segments is to be taken, as they have contrary signs affirmative and negative, the difference of the quantities is that sum.—Súr. See *Vij-gan.* § 5.

[1] For finding the gross area of a quadrilateral, and, by extension of the rule, the exact area of a trangle.—Gan. For finding the area by a method delivered by Sridhara—Rang.

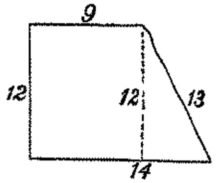
[2] If the three remainders be added together, their sum is equal to half the sum of all the sides. The product of the continued multiplication of the three remainders being taken into the sum of those remainders, the product so obtained is equal to the product of the square of the perpendicular taken into the square of half the base. [*It* is not explained how this is the case. The last mentioned product $=$

$\frac{1}{4}a^2\left\{b^2-\frac{1}{4}\left(a+\frac{b^2-c^2}{a}\right)^2\right\}$, (taking a as base, and supposing $b > c$, and applying §163—164) $=\frac{4a^2b^2-(a^2+b^2-c^2)^2}{16}=\frac{1}{16}(a+b+c)(a+b-c)(a-b+c)(b+c-a)=s(s-a)(s-b)(s-c)=(s-a)(s-b)(s-c)(s-a+s-b+s-c.)$—Ed.] It is a square quantity; for a square multiplied by a square gives a square. The square root being extracted, the product of the perpendicular by half the base is the result; and that is the area of the triangle. Therefore the true area is thus found. In a quadrilateral, the product of the multiplication does not give a square quantity, but an irrational one. Its approximate root is the area of the figure; not, however, the true one: for, when divided by the perpendicular, it should give half the sum of the base and summit.—Súr. [The last remark does not hold good unless the quadrilateral be a trapezium.—Ed.]

four sides alone without an angle, whereas a triangle is determined by its three sides. Hence it is incorrect to say that the expression derived from the above rule represents the gross area of a quadrilateral. See note to §§169—170. The proof of the rule given by Súryadása is not at all clear. See foot-note.]

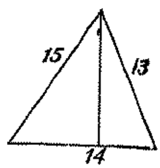
168. Example. In a quadrilateral figure, of which the base[1] is fourteen, the summit[2] nine, the flanks thirteen and twelve, and the perpendicular twelve, tell the area as it was taught by the ancients.

Statement: base 14; summit 9; sides 13 and 12; perpendicular 12. By the method directed, the result obtained is the surd 19800, of which the approximate root is somewhat less than 141. That, however, is not in this figure the true area. But, found by the method which will be set forth (§ 175), the true area is 138.



Statement of the triangle before instanced (§ 165).

By the (present) method the area comes out the same, *viz.*, 84.



169—170. Aphorism comprised in a stanza and a half. Since the diagonals of the quadrilateral are indeterminate, how should the area be in this case determinate? The diagonals found as assumed by the ancients[3] do not answer in another case. With the

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[1] The greatest of the four sides is called the base —Gan. This definition is, however, too restricted. See §§178, 185.

[2] *Mukha*, *vadana*, or any other word denoting mouth; the side opposite to the base, the summit.

[3] By Srídhara and the rest.—Gan.

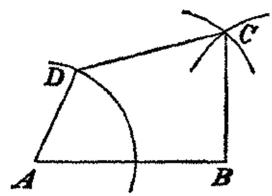
same sides, there are other diagonals ; and the area of the figure is accordingly manifold.

For, in a quadrilateral, opposite angles being made to approach, contract their diagonal as they advance inwards : while the other angles receding outwards lengthen their diagonal. Therefore it is said that with the same sides there are other diagonals.

171. How can a person, neither specifying one of the perpendiculars, nor either of the diagonals, ask the rest?[1] Or how can he demand a determinate area, while they are indefinite?

172. Such a questioner is a blundering devil.[2] Still more so is he, who answers the question. For he considers not the indefinite nature of the lines[3] in a quadrilateral figure.

[The four sides alone without an angle do not determine the quadrilateral in general, and the area is consequently indeterminate. For, take AB as one side ; and with centre A and radius equal to another side describe a circle ; take *any point* D in the circumference, and join AD. With D, B as centres and radii equal to the other two sides, describe circles cutting each other in C. Join CB, CD. Then $ABCD$ is the quadrilateral which is indeterminate, since the angle BAD being not given, the point D may be taken anywhere in the circumference of the first described circle. Hence with the same sides, the diagonals may vary. But if the perpendicular from D to the line AB be given, or if the diagonal BD be given, it is easy to see that the point D becomes a fixed point in the circumference of the first de-



[1] The perpendiculars, diagonals, &c.—Gan.
[2] *Pisácha*, a demon or vampire ; so termed because he blunders.—Súr.
[3] Of the diagonal and perpendicular lines.—Súr.

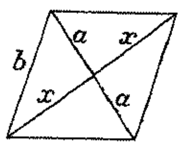
scribed circle, and so the quadrilateral is determinate. In the case of a trapezium, it is easily seen that the four sides being given, the distance between two paralled sides is known, and so the figure is determinate, The rule in §167, however, is equally inapplicable to this case.]

173—175. Rule[1] in two and a half stanzas. Let one diagonal of an equilateral tetragon be put as given. Then subtract its square from four times the square of the side. The square root of the remainder is the measure of the second diagonal.

The product of unequal diagonals multiplied together, being divided by two, will be the precise area in an equilateral tetragon. In a regular one with equal diagonals, as also in an oblong,[2] the product of the side and upright will be so.

In any other quadrilateral with equal perpendiculars, the moiety of the sum of the base and summit, multiplied by the perpendicular (is the area).

[In an equilateral tetragon, the diagonals bisect each other at right angles. Hence (see figure), $x=\sqrt{b^2-a^2}$;
$\therefore\ 2x=$ unknown diagonal $=\sqrt{4b^2-4a^2}$.



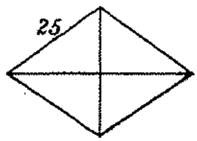
The area of the above figure is evidently equal to half the product of the diagonals. The other propositions stated above are well known elementary geometrical results. By "a quadrilateral with equal perpendiculars," the author means a trapezium.

[1] In an equilateral tetragon, one diagonal being given, to find the second diagonal and the area; also in an equiperpendicular tetragon (trapezium) to find the area.—Gan. Equilateral tetragons are two-fold: with equal and with unequal diagonals. The first rule regards the equilateral tetragon with unequal diagonals (the rhombus).—Súr.

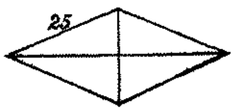
[2] *Áyata*, a long quadrilateral which has pairs of equal sides.—Gan.

176. Example.. Mathematician, tell both diagonals and the area of an equilateral quadrangular figure whose side is the square of five: and the area of it, the diagonals being equal: also (the area) of an oblong, the breadth of which is six and length eight.

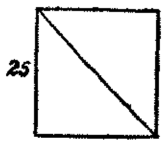
Statement of the first figure (rhombus). Here, assuming one diagonal 30, the other is found 40; and the area is 600.



Or put one diagonal 14; the other is found 48; and the area is 336. See figure.



Statement of the second figure (square).



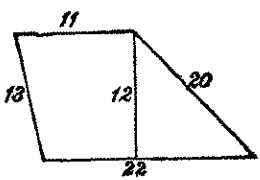
Here, taking the square root of the sum of the squares (§ 134), the diagonal comes out as the surd $\sqrt{1250}$, alike both ways. The area is 625.

Statement of the third figure (oblong). Area 48.

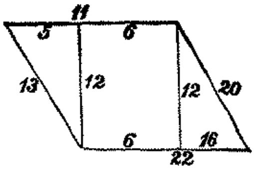
177. Example. Where the summit is eleven, the base twice as much as the summit, the flanks thirteen and twenty, and the perpendicular twelve; say what the area will be.

Statement:

The gross area (§ 167) is 250. The true area (§175) is 198.



Or making three portions of the figure, and severally finding their areas, we get 30, 72, 96 (see figure); and summing up we get for the total area 198 as before.



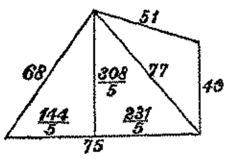
[The four sides of the trapezium being given, the perpendicular is necessarily known.]

178. Another example. Declare the diagonal, perpendicular and dimensions of the area, in a figure of which the summit is fifty-one, the base seventy-five, the left side sixty-eight, and the other side twice twenty.

179. Aphorism showing the connection of area, perpendicular and diagonal. If the perpendicular be known, the diagonal is so; if the diagonal be known, the perpendicular is so. If they be definite, the area is determinate. For, if the diagonal be indefinite, so is the perpendicular. Such is the meaning.

179 *continued.* Rule for finding the perpendicular.[1] In the triangle within the quadrilateral, the perpendicular is found as before taught (§163—164); the diagonal and side being sides, and the base, a base.

Here, to find the perpendicular, a diagonal proceeding from the extremity of the left side to the origin of the right one is assumed



to be 77; see figure. By this a triangle is constituted within the quadrilateral. In it that diagonal is one side, 77; the left side is another, 68; the base continues such, 75. Then, proceeding by the rule (§§163—164),

[1] The diagonal being either given or assumed.—Gan.

the segments are found, $\frac{144}{5}$ and $\frac{231}{5}$; and the perpendicular, $\frac{308}{5}$.

[The problem given is indeterminate, unless a diagonal, or an angle, or a perpendicular distance be given. So one diagonal is supposed to be 77. The process then adopted is the same as that shown in the note to §§163—164.]

180. Rule to find the diagonal, when the perpendicular is known.

The square root of the difference of the squares of the perpendicular and its adjoining side is pronounced the segment. The square of the base less that segment being added to the square of the perpendicular, the square root of the sum is the diagonal.

In the above quadrilateral, the perpendicular from the extremity of the left side is put $\frac{308}{5}$. Hence the segment is found $\frac{144}{5}$; and by the rule (§180) the diagonal comes out 77.

[The reason for the rule is manifest.]

181—182. Rule to find the second diagonal: two stanzas.

In this figure, first a diagonal is assumed.[1] In the two triangles situated one on each side of the diagonal, this diagonal is made the base of each; and the other sides are given: the perpendiculars and segments[2] must be found. Then the square of the difference of two segments on the same side[3] being added to the square of the sum of the perpendiculars, the square root of

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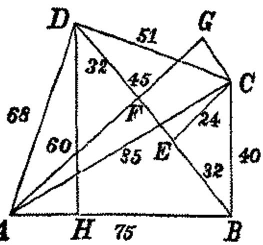
[1] Either arbitrarily (see §183) or as given by the conditions of the question.—Gan.

[2] The two perpendiculars and the four segments.—Gan.

[3] Square of the interval of two segments measured from the same extremity.

the sum of those squares will be the second diagonal in all tetragons.[1]

In the same quadrilateral, the length of the diagonal passing from the extremity of the left side to the origin of the right one, is put 77. Within the figure cut by that diagonal line, two triangles are formed, one on each side of the diagonal. Taking the diagonal for the base of each, and the two other sides as given, the two perpendiculars and the several segments must be found by the method before taught. Thus the perpendiculars are found, 24 and 60. Segments of the base made by the former, 45 and 32; those made by the latter, 32 and 45. Difference of the segments on the same side (that is, so much of the base as is intercepted between the perpendiculars) is 13. Its square 169. Sum of the perpendiculars 84. Its square 7056. Sum of the squares 7225. Square root of the sum 85. It is the length of the second diagonal. So in every like instance.



[1] In the figure which is divided by the diagonal line, two triangles are contained, one on each side of that line; and their perpendiculars, which fall one on each side of the diagonal, are thence found. The difference between two segments on the same side will be the interval between the perpendiculars. It is taken as the upright of a triangle. Producing (see above figure) one perpendicular by the addition of the other, (*i. e.*, drawing *CG* perpendicular to *AF* produced), the sum (*AG*) is made the side of the triangle. The second diagonal (*AC*) is hypotenuse. A triangle (*AGC*) is thus formed. From this is deduced, that the square root of the sum of the squares of the upright (which $= CG = EF = BF - BE$) and side (which $= AG = AF + CE$) will be the second diagonal: and the rule is demonstrated.—Gan.

In an equilateral tetragon, there is no interval between the perpendiculars; and the second diagonal is the sum of the perpendiculars.—*Ibid.*

[The reason for the rule will be clear from the explanation given by Ganesa, cited in the foot-note. In the particular example chosen, it happens (see above figure) that $DF=EB$, but this need not always be the case. It also happens from the values of AB, BC, CA, that the angle ABC is a right angle.]

183. Rule restricting the arbitrary assumption of a diagonal: a stanza and a half. The sum of the shortest pair of sides containing the diagonal being taken as a base, and the remaining two as the legs (of a triangle), the perpendicular is to be found: and, in like manner, with the other diagonal. The diagonal cannot by any means be longer than the corresponding base, nor shorter than the perpendicular answering to the other. Adverting to these limits, an intelligent person may assume a diagonal.

For a quadrilateral, contracting as the opposite angles approach, becomes a triangle; wherein the sum of the least pair of sides about one angle is the base, and the other two are taken as the legs. The perpendicular is found in the manner before taught. Hence the shrinking diagonal cannot by any means be less than the perpendicular; nor the other be greater than the base. It is so both ways. This, even though it were not mentioned, would be readily perceived by the intelligent student.

[What the author intends to say is (see figure to §§ 181-182) that the diagonal BD cannot be longer than $DC+BC$, but always shorter; nor can it be shorter than the perpendicular DH, but always longer. (Euclid, I. 20 and 19). The first sentence of the above section is meaningless; and so also is the proof given, *viz.*, "For a quadrilateral, contracting, &c."]

184. Rule to find the area: half a stanza. The

sum of the areas of the two triangles on either side of the diagonal is assuredly[1] the area in this figure.

In the figure last specified, the areas of the two triangles are 924 and 2310. The sum of these is 3234, the area of the tetragon.

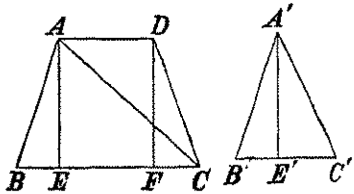
[The area of the triangle BCD (see figs. to §§ 181-182) is 924, and that of the triangle ABD is 2310.]

185—186. Rule: two stanzas. Making the difference between the base and summit of a (trapezium or) quadrilateral that has equal perpendiculars, the base (of a triangle), and the sides (its) legs, the segments of it and the length of the perpendicular are to be found as for a triangle. From the base of the trapezium subtract the segment, and add the square of the remainder to the square of the perpendicular; the square root of the sum will be the diagonal.

In a (trapezium or) quadrilateral that has equal perpendiculars, the sum of the base and least flank is greater than the aggregate of the summit and other flank.

[The rule gives the method of finding the diagonals of a trapezium the sides of which are given. It is demonstrated by Ganesa in the following manner:—

Let the two triangles ABE, DFC be united into one triangle $A'B'C'$, their altitudes being equal. Then the altitude $A'E'$ of the new triangle $A'B'C'$ is the altitude of the trapezium, and the segments $B'E'$, $E'C'$ will be equal to BE, FC. Hence $AC^2 = AE^2 + EC^2 = A'E'^2 + (BC - B'E')^2$, which leads to the rule.



[1] It is the true and correct area, contrasted with the gross or inexact area of former writers.—Gan. and Súr.

Also, $A'B'+B'C'>A'C'$,

$\therefore AB+BE+FC+EF>AD+DC$, $(\because AD=EF)$

i. e., $AB+BC \quad >AD+DC$.

AB need not be the 'least flank.']

187—189. Example. The sides measuring fifty-two and one less than forty, the summit equal to twenty-five, and the base sixty, this was given as an example by former writers for a figure having unequal perpendiculars; and definite measures of the diagonals were stated, fifty-six and sixty-three. Assign to it other diagonals, and those particularly which appertain to it as a figure with equal perpendiculars.

Statement.

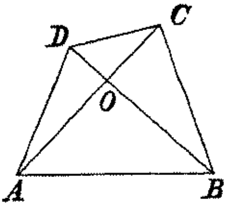


Fig. 1.

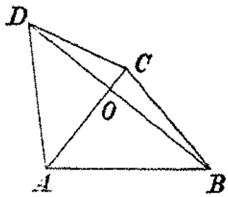


Fig. 2.

Here assuming one diagonal 63, the other is found as before, 56. Or, putting 32 instead of 56 for a diagonal (Fig. 2), the other, found by the process before shown, comes out in two portions, both surds, $\sqrt{621}$ and $\sqrt{2700}$. The sum of the roots (extracted by approximation) is the second diagonal 76 $\frac{22}{25}$.

[In Fig. 2, if we drop perpendiculars from B, D on AC, and find the segments of AC by § 163, we shall find that AC, BD intersect at right angles, and that $AO=30$.

Hence $BD=BO+DO=\sqrt{60^2-30^2}+\sqrt{39^2-30^2}$
$=\sqrt{2700}+\sqrt{621}=30\sqrt{3}+3\sqrt{69}=30\times1{\cdot}732...+3\times8{\cdot}307...$
$=51{\cdot}96...+24{\cdot}921...=76{\cdot}88...=76\frac{22}{25}$ nearly. The root is approximately extracted in the manner indicated in § 138.]

Again, if the above quadrilateral (Fig. 1) be one with equal perpendiculars, *i.e.*, a trapezium, (Fig. 3), consider the triangle $A'B'D'$ (Fig. 4), put to find the perpendicular DE, and the segments AE, FB, according to the rule in §§ 185-186. Here the segments are found $\frac{3}{5}$ and $\frac{172}{5}$; and the perpendicular, the surd $\sqrt{\frac{38016}{25}}$, of which the root found by approximation is $38\frac{622}{625}$. It is the equal perpendicular of the trapezium.

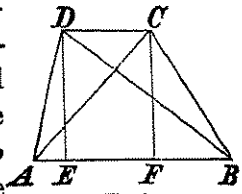


Fig. 3.

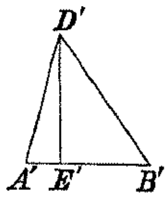


Fig. 4.

Next to find the sum of the squares of the perpendicular and difference between base and segment, we have, base of trapezium, 60; least segment $\frac{3}{5}$; difference $\frac{297}{5}$; square of the difference $\frac{88209}{25}$; square of the perpendicular $\frac{38016}{25}$; sum $\frac{126225}{25}$, or dividing by the denominator, 5049. It is the square of one diagonal (BD). Similarly, the square of the other diagonal (AC) is 2176. Extracting the roots of these squares by approximation, the two diagonals come out $71\frac{1}{25}$ and $46\frac{16}{25}$.

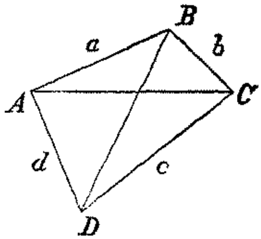
In the above trapezium, the short side 39 added to the base 60 makes 99, which is greater than the aggregate of the summit and other flank, 77. Such is the limitation.

Thus, with the same sides, may be various diagonals in the tetragon. Yet, though indeterminate, diagonals have been sought as determinate, by Brahmagupta and others. Their rule is as follows :—

190. **Rule.**[1] The sums of the products of the sides about both the diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square roots of the results are the diagonals in a quadrilateral.

The objection to this mode of finding the diagonals is its operoseness, as I shall show by proposing a shorter method.

[The rule applies only to a quadrilateral which can be inscribed in a circle. This, however, is not mentioned in the rule.



Let $ABCD$ be such a quadrilateral, and let $AB=a$, $BC=b$, $CD=c$, $DA=d$.

Then $AC^2=\frac{(ac+bd)(ad+bc)}{ab+cd}$,

and $BD^2=\frac{(ac+bd)(ab+cd)}{ad+bc}$.

(See Todhunter's Trigonometry, Art. 254.) Thus the reason for the rule is obvious.

In the quadrilateral in §§ 187-189, Fig. 1, the sides are so taken that the diagonals intersect at right angles; and we easily find $OB=48$, $OD=15$. Thus $\cos(ODA)=\frac{15}{39}=\frac{5}{13}$, and $\cos(OCB)=\frac{20}{52}=\frac{5}{13}$. Hence angle ODA=angle OCB, and therefore a circle passes round the quadrilateral. Consequently, the rule in § 190 will apply to this quadrilateral. It is probable that the rule was derived *a posteriori* from this particular instance, and not *a priori* from the fact of the quadrilateral being inscribable in a circle. The same quadrilateral is given as an example of the rule by Chaturveda Prithúdaka Swámí, in his commentary on Brahmagupta's treatise.]

191—192. Rule : two stanzas. The uprights and

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[1] A couplet cited from Brahmagupta, XII, 28.

sides of two assumed right-angled triangles,[1] being multiplied by the reciprocal hypotenuses, become sides (of a quadrilateral): and in this manner is constituted a quadrilateral, in which the diagonals are deducible from the two triangles.[2] The product of tne uprights added to the product of the sides is one diagonal; the sum of the products of uprights and sides reciprocally multiplied, is the other.[3] When this short method exists, why an operose one was practised by former writers, we know not.

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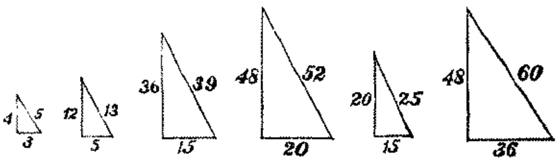
[1] Assumed conformably with the rule in §145. An objection, to which Ganesa adverts and which he endeavours to obviate, is that this shorter method requires sagacity in the selection of assumed triangles; whereas the longer method is adapted to all capacities.

[2] This method of constructing a quadrilateral is taken from Brahmagupta, XII, 38.

[3] A quadrilateral is divided into four triangles by its intersecting diagonals; and conversely, by the junction of four triangles, a quadrilateral is constituted. For that purpose, four triangles are assumed in this manner. Two triangles are first put in the mode directed (§145), the sides of which are all rational. Such sides, multiplied by any assumed number, will constitute other right-angled triangles, of which also the sides will be rational. By the twofold multiplication of hypotenuse, upright and side of one assumed triangle by the upright and side of the other, four (right-angled) triangles are formed, such that turning and adapting them and placing the multiples of the hypotenuses for sides, a quadrilateral is composed, (as shown below).



Here the uprights and sides of the arbitrary triangles (the first two on the left side), reciprocally multiplied by the hypotenuses, become sides of the quadrilateral; and hence the directions of the rule (§191).

In a quadrilateral so constituted, it is apparent that the one diagonal (AC) is composed of two parts; one the product of

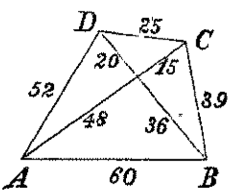
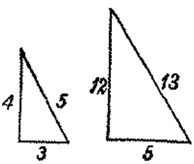


Fig. 1.

[This rule is of no importance whatever. It applies only to quadrilaterals constructed in the artificial manner indicated by the author, and fully explained by Ganesa ; see foot-note. It is curious that the author, while censuring Brahmagupta's important rule (§190) as operose, entirely forgets that the rule propounded by himself is comparatively unimportant and of very limited application.]

Assuming two right-angled triangles, multiply the upright and side of one by the hypotenuse of the other : the greatest of the products is taken for the base ; the least for the summit ; and the other two for the flanks. (See Fig. 1 in the foot-note.)



the uprights, the other the product of the sides of the arbitrary triangles. The other diagonal (BD) consists of two parts, *viz.*, the products of the reciprocal multiplication of uprights and sides. These two portions are the perpendiculars, for there is no interval between the points of intersection. This holds, provided the shortest side be the summit; the longest, the base; and the rest, the flanks. But if the component triangles be otherwise adapted, the summit and a flank change places, as in the adjoining figure. Here the two portions of the first diagonal as above found (*viz.*, 48 and 15) do not face, but are separated by an interval, which is equal to the difference between the two portions (36 and 20) of the other diagonal, *viz.*, 16. It is the difference of two segments on the same side, found by a preceding rule (§§181-182), and is taken for the upright of a triangle as already explained (§§181-182, note); the sum of the two portions of the diagonal equal to the two perpendiculars is made the side. The square root of the sum of the squares of such upright and side is equal to the product of the hypotenuses (13 and 5): wherefore the author adds, "if the summit and flank change places, the *first* diagonal will be the product of the hypotenuses." (The MSS. have *first*, but Bháskara's text exhibits *second* instead.)

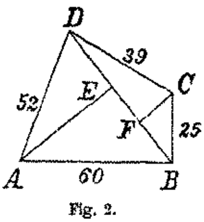


Fig. 2.

[This last remark is not clearly explained by the commentator. From the calculated values of AE and FB (see Fig. 2), we find that sin $ABE = \frac{AE}{AB} = \frac{48}{60} = \frac{4}{5}$; and cos $CBF = \frac{FB}{BC} = \frac{20}{25} = \frac{4}{5}$. Hence the angle ABC is a right angle

Here with much labour (by the former method) the diagonals are found 63 and 56.

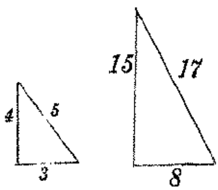
With the same pair of right-angled triangles, the products of uprights and sides reciprocally multiplied are 36 and 20; the sum of which is one diagonal, 56. The products of uprights multiplied together, and sides taken into each other, are 48 and 15; their sum is the other diagonal, 63. Thus they are found with ease.

But if the summit and flank change places, and the figure be stated accordingly, the second diagonal will be the product of the hypotenuses of the two right-angled triangles, *viz.*, 65. (See Fig. 2 in the foot-note.)

And ∵ $AB = 5 \times 12$, and $BC = 5 \times 5$,

∴ $AC = 5 \times 13$, and the truth of the remark is obvious. It need hardly be added that the remark applies only to quadrilaterals constructed in the artificial manner indicated by the author.—Ed.]

In like manner, for the tetragon before instanced (§178), to find the diagonals, a pair of rectangular triangles is put. Proceeding as directed, the diagonals come out 77 and 84. (See Fig. 3.)



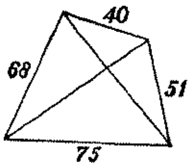


Fig. 3.

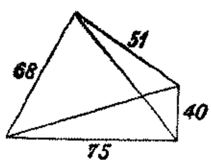
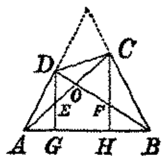


Fig. 4.

In the figure instanced, a transposition of the flank and summit takes place (see Fig. 4 which corresponds to the tetragon instanced in §178); wherefore the product of the hypotenuses (5 and 17) of the two right-angled triangles will be the second diagonal; and they thus come out 77 and 85.—Gan.

193—194. Example.[1] In a figure in which the base is three hundred, the summit a hundred and twenty-five, the flanks two hundred and sixty and one hundred and ninety-five, one diagonal two hundred and eighty and the other three hundred and fifteen, and the perpendiculars a hundred and eighty-nine and two hundred and twenty-four, what are the portions of the perpendiculars and diagonals below the intersections of them ? and the perpendicular let fall from the intersection of the diagonals, with the segments answering to it ? and the perpendicular of the needle formed by the prolongation of the flanks until they meet, as well as the segments corresponding to it ? and the measure of both the needle's sides ? All this declare, mathematician, if thou be thoroughly skilled in this (science of[2]) plane figure.

Statement. Length of the base 300. Summit 125. Flanks 260 and 195. Diagonals 280 and 315. Perpendiculars 189 and 224.



195—196. Rule: two stanzas. The interval between the perpendicular and its correspondent flank is termed the *sandhi*[3] or link of that perpendicular. The

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[1] Having thus, from §173 to this place, shown the method of finding the area, &c., in the fourteen sorts of quadrilaterals, the author now exhibits another quadrilateral, proposing questions concerning segments produced by intersection.—Gan. For the instruction of the pupil, he exhibits the figure called (*súchi*) a needle.—*Mano.*

The problem is taken from Brahmagupta with a slight variation ; and this example differs from his only in the scale, his numbers being here increased five-fold. See Brahmagupta, XII, 32.

[2] *Manoranjana.*

[3] *Sandhi*, union, alliance ; connecting link.

base less the link or segment is called the *píthá*[1] or complement of the same. The link or segment contiguous to that portion (of perpendicular or diagonal) which is sought, is twice set down. Multiplied by the other perpendicular in one instance, and by the diagonal in the other, and divided (in both instances) by the complement belonging to the other (perpendicular), the quotients will be the lower portions of the perpendicular and diagonal below the intersection.

Statement. Perpendicular 189. Flank contiguous to it 195. Segment intercepted between them (found by §134), 48. It is the link. The second segment is 252, and is called the complement.

In like manner, the second perpendicular is 224. The flank contiguous to it, 260. Interval between them, being the segment called link, 132. Complement 168.

Now to find the lower portion of the first perpendicular 189. Its link 48 separately multiplied by the other perpendicular 224 and by the diagonal 280, and divided by the other complement 168, gives quotients 64, the lower portion of the perpendicular, and 80, the lower portion of the diagonal.

So for the second perpendicular 224, its link 132, severally multiplied by the other perpendicular 189 and by the diagonal 315, and divided by the other complement 252, gives 99 for the lower portion of the perpendicular, and 165 for that of the diagonal.

[The object of the rule is to find EG, EA, FH, FB. (See figure, §§193-194).

From similar triangles we have,

$$\frac{EG}{GA}=\frac{CH}{HA}, \text{ whence } EG=\frac{GA\times CH}{HA};$$

---

[1] *Pítha*, lit. stool. Here the complement of the segment.

and $\frac{AE}{AC}=\frac{GA}{HA}$, whence $AE=\frac{GA\times AC}{HA}$

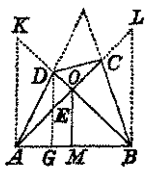
Similarly for FH and FB.

Hence the reason for the rule is clear.]

197. Rule to find the perpendicular below the intersection of the diagonals.

The perpendiculars multiplied by the base and divided by the respective complements, are the erect poles : from which the perpendicular let fall from the intersection of the diagonals, as also the segments of the base, are to be found as before.[1]

Statement. Proceeding as directed, the erect poles are found 225 and 400. Whence, by a former rule (§159), the perpendicular below the intersection of the diagonals is deduced, 144 ; and the segments of the base 108 and 192.



[The method employed is first to find AK, BL, the erect poles or perpendiculars on AB, and then to apply §159 to find OM. Now from similar triangles we have $\frac{AK}{AB}=\frac{GD}{GB}$, whence $AK=\frac{GD\times AB}{GB}$, with a similar value for BL. Hence the rule.]

198—200. Rule to find the perpendicular of the needle,[2] its legs and the segments of its base : three stanzas. The proper link multiplied by the other perpendicular and divided by its own, is termed the mean ;[3] and the sum of this and the opposite link is called the divisor.[4] Those two quantities, namely, the mean and

[1] By the rule in §159.

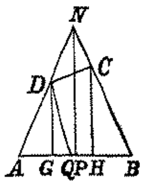
[2] *Súchí*, needle ; the triangle formed by the flanks of the quadrilateral until they meet.

[3] *Sama*, mean ; a fourth proportional to the two perpendiculars and the link or segment.

[4] *Hara*, divisor ; the sum of the *mean* and the other link or segment.

the opposite link, being multiplied by the base and divided by that divisor, will be the respective segments of the needle's base. The other perpendicular, multiplied by the base and divided by the divisor, will be the perpendicular of the needle. The flanks, multiplied by the perpendicular of the needle and divided by their respective perpendiculars, will be the legs of the needle. Thus may the subdivision of a plane figure be conducted by the intelligent, by means of the Rule of Three.

Here the perpendicular being 224, its link is 132. This multiplied by the other perpendicular, *viz.*, 189, and divided by its own, *viz.*, 224, gives the mean as it is named, $\frac{891}{8}$. The sum of this and the other link 48 is the divisor as it is called, $\frac{1275}{8}$. The mean and the other link severally taken into the base, being divided by this divisor, give the segments of the needle's base, $\frac{1536}{17}$ and $\frac{3564}{17}$. The other perpendicular 189, multiplied by the base and divided by the same divisor, yields the perpendicular of the needle, $\frac{6048}{17}$. The sides 195 and 260, multiplied by the needle's perpendicular and divided by their own perpendiculars respectively, *viz.*, 189 and 224, give the legs of the needle, which are the sides of the quadrilateral produced, *viz.*, $\frac{6240}{17}$ and $\frac{7020}{17}$.



Thus in all instances under this head, taking the divisor for the argument, and making the multiplicand or multiplicator, as the case may be, the fruit or requisition, the Rule of Three is to be inferred by the intelligent mathematician.

[The object of the rule is to find PA, PB, NP, NA, NB (see above figure). It is demonstrated by Ganesa in the following manner :—Through D draw DQ parallel to NB, meeting AB in Q. Then the triangles AQD and ABN are similar, and we evidently get $\frac{AB}{AQ}=\frac{AN}{AD}=\frac{NP}{DG}=\frac{PB}{GQ}=\frac{PA}{GA}$. . . . (1). Now $\frac{GQ}{DG}=\frac{HB}{CH}$ from similar triangles ; thus $GQ=\frac{HB\times DG}{CH}$ = what the author calls *mean*, and consequently AQ or *mean* $+AG=$ what the author calls *divisor*. Thus finally, $PB=\frac{GQ\times AB}{AQ}$ from (1) $=\frac{\textit{mean}\times\text{base}}{\textit{divisor}}$; and $PA=\frac{GA\times AB}{AQ}$ from (1) $=\frac{\text{opposite link}\times\text{base}}{\textit{divisor}}$. Again, from (1) $\frac{NP}{DG}=\frac{AB}{AQ}$, whence $NP=\frac{\text{other perpendicular}\times\text{base}}{\textit{divisor}}$. Lastly, $\frac{NA}{DA}=\frac{NP}{DG}$, whence $NA=\frac{DA\times NP}{DG}$, and similarly for NB.]

201. Rule.[1] When the diameter of a circle[2] is multiplied by three thousand nine hundred and twenty-seven and divided by twelve hundred and fifty, the quotient is the near[3] circumference : or multiplied by twenty-two and divided by seven, it is the gross circumference adapted to practice[4].

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[1] To deduce the circumference of a circle from its diameter, and the diameter from the circumference.—Gan.

[2] *Vritta, vartula*, a circle. *Vyása, vishkambha, vistriti, vistára*, the breadth or diameter of a circle. *Paridhi, parináha, vritti, nemi* (and other synonyms of the felloe of a wheel), the circumference of a circle.

[3] *Súkshma*, delicate or fine, nearly precise ; contrasted with *sthúla*, gross, or somewhat less exact, but sufficient for common purposes.—Gang. and Súr.

Brahmagupta puts the ratio of the circumference to the diameter as three to one for the gross value, and takes the square root of ten times the square of the diameter for the neat value of the circumference. See Brahma. XII, 40. [This is more rough even than $\frac{22}{7}$; for $\sqrt{10}=3{\cdot}1622\ldots$, and $\frac{22}{7}=3.\dot{1}4285\dot{7}$.—Ed.]

[4] As the diameter increases or diminishes, so does the circumference increase or diminish : therefore to find the one from the other, make proportion, as the diameter of a known circle is to the known circumference, so is the given diameter to the circumference sought : and conversely, as the

[Ganesa shows (see foot-note) that if the measure of the diameter of a circle be 1250, that of the side of a regular polygon of 384 sides inscribed in the circle will be very nearly 3927 (more accurately it will be $= \sqrt{98683} \times 12{\cdot}5 = 3926{\cdot}625 \ldots\ldots$). This shows the degree of approximation of the fraction $\frac{3927}{1250}$ to the value of π. Converting the fraction into a decimal we get 3·1416, the true value of π being 3·14159 . . . .]

202. Example. Where the measure of the diameter is seven, friend, tell the measure of the circumference: and where the circumference is twenty-two, find the diameter.

Statement : diameter 7.

Answer : circumference $21\frac{1239}{1250}$, or gross circumference 22.

Statement : circumference 22.

Reversing multiplier and divisor, the diameter comes out $7\frac{11}{3927}$; or gross diameter 7.

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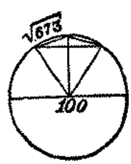
circumference is to the diameter, so is the given circumference to the diameter sought.

Further, the semi-diameter is equal to the side of a regular hexagon within the circle, as will be shown. From this the side of a regular dodecagon may be found in this manner :—the semi-diameter being hypotenuse, and half the side of the hexagon, the side, the square root of the difference of their squares is the upright: subtracting which from the semi-diameter, the remainder is the arrow (or height of the arc). Again, this arrow being the upright and half the side of the hexagon, a side, the square root of the sum of their squares is the side of the dodecagon. From this, in like manner, may be found the side of a polygon of 24 sides: and so on, doubling the number of sides in the polygon, until the side be near to the arc. The sum of such sides will be the circumference of the circle nearly. Thus, the diameter being 100, the side of the dodecagon is the surd $\sqrt{673}$; and that of a polygon of 384 sides is nearly equal to the arc. By computation it comes out the surd $\sqrt{98683}$. Now the proportion, if to the square of the diameter put at 100, *viz.*, 10000, this be the square of the circumference, *viz.*, 98683, then to the square of the assumed diameter 1250, *viz.*, 1562500, what will be the circumference? Answer : the square root 3927 without remainder.—Gan.



203. Rule. **In a circle, a quarter of the diameter multiplied by the circumference is the area. That multiplied by four[1] is the net all around the ball.[2] This content of the surface of the sphere, multiplied by the diameter and divided by six, is the precise solid, termed cubic, content within the sphere.[3]**

[This rule gives the well-known expressions for the area of a circle, and the surface and volume of a sphere. Ganesa gives

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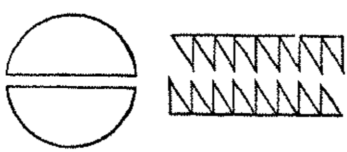
[1] Or rejecting equal multiplier and divisor, the circumference multiplied by the diameter is the surface.—Gan.

[2] *Prishtha-phala*, superficial content: compared to the net formed by the string with which cloth is tied to make a playing ball. *Ghana-phala*, solid content: compared to a cube, and denominated from it, cubic.



[3] Dividing the circle into two equal parts, cut the content of each into any number of equal angular spaces, and expand it so that the circumference may become a straight line as in the adjoining figure. Then let the two portions approach so that the angular spaces of the one may enter into the similar intermediate vacant spaces of the other, as in the figure, thus constituting an oblong, of which the semi-diameter is one side, and half the circumference the other. The product of their multiplication is the area. Half by half is a quarter. Therefore a quarter of the diameter multiplied by the circumference is the area.—Gan.

See in the *Goládhyáya* of the *Siddhánta-siromani*, a demonstration of the rule, that the surface of a sphere is four times the area of a great circle, or equal to the circumference multiplied by the diameter.—*Ibid.* [See the *Goládhyáya*, Wilkinson's translation, III, 52.—Ed.]

To demonstrate the rule for the solid content of a sphere, suppose the sphere divided into as many little pyramids, or long needles with an acute tip and square base, as is the number by which the surface is measured, the height of each pyramid being equal to the radius of the sphere. The base of each pyramid is a unit of the scale by which the dimensions of the surface are reckoned; and the altitude being a semi-diameter, one-third of their product is the content: for a needle-shaped excavation is one-third of an excavation in the form of a rectangular parallelopiped of the same base and height, as will be shown (§221). Therefore (unit taken into) a sixth part of the diameter is the content of one such pyramidical portion: and that multiplied by the surface gives the solid content of the sphere.—*Ibid.*

interesting but rough demonstrations in the case of the area of a circle and the volume of a sphere, and refers to the *Goládhyáya* for the case of the surface of a sphere. See foot-note.]

204. Example. Intelligent friend, if thou know well the spotless *Lílávatí*, say what the area of a circle is, the diameter of which is measured by seven : and the surface of a globe, or area like a net upon a ball, the diameter being seven: and the solid content within the same sphere.

Statement: diameter 7.

Answer: area of the circle, $38\frac{2423}{5000}$. Superficial content of the sphere, $153\frac{1173}{1250}$. Solid content of the sphere, $179\frac{1487}{2500}$.

205—206. Rule : a stanza and a half. The square of the diameter being multiplied by three thousand nine hundred and twenty-seven, and divided by five thousand, the quotient is the nearly precise area ; or multiplied by eleven and divided by fourteen, it is the gross area adapted to common practice. Half the cube of the diameter, with its twenty-first part added to it, is the solid content of the sphere.

The area of the circle, nearly precise, comes out as before $38\frac{2423}{5000}$, or gross area $38\frac{1}{2}$. Gross solid content $179\frac{2}{3}$.

[The area of a circle $=\frac{1}{4}\pi d^2$, d being the diameter

$$=\frac{3927}{4\times 1250}d^2 = \frac{3927}{5000}d^2;$$

or more roughly $=\frac{22}{4\times 7}d^2 \quad = \frac{11}{14}d^2$.

Again, the volume of a sphere $= \frac{4}{3}\pi\frac{d^3}{8}$, d being the diameter

$$=\frac{22}{7\times 6}d^3 = \frac{11}{21}d^3 \text{ roughly}$$

$$=\frac{1}{2}d^3\left(1+\frac{1}{21}\right).$$

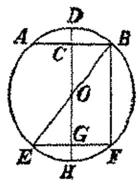
Thus the reasons for the rules are obvious.]

206—207. Rule[1] : a stanza and a half. The sum and difference of the chord and diameter being multiplied together, and the square root of the product being subtracted from the diameter, half the remainder is the arrow.[2] The diameter less the arrow being multiplied by the arrow, twice the square root of the product is the chord. The square of half the chord being divided by the arrow, the quotient added to the arrow is pronounced to be the diameter of the circle.

[By the word *arrow* is meant the height of the arc.

Súryadása gives the following proof of the formula for the arrow.

Let AB be the chord, CD, the arrow, and O the centre. Join BO, and produce it to meet the circumference in E. Draw the chord EGF perpendicular to the diameter DH. Join BF.



Then $CD = \frac{1}{2}(DH - CG) = \frac{1}{2}(DH - BF)$
$= \frac{1}{2}(DH - \sqrt{BE^2 - EF^2}) = \frac{1}{2}(DH - \sqrt{DH^2 - AB^2})$,
whence the rule.

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[1] In a circle cut by a right line, to find the chord, arrow, &c.; that is, either the chord, the arrow, or the diameter being unknown, and the other two given, to find the one from the others.—Gan. and Súr.

[2] A portion of the circumference is a bow (*dhanush, chápa*). The right line between its extremities, like the string of a bow is its chord (*jívá, jyá, guna, maurví*). The line between them is the arrow (*sara, ishu*), as resembling one set on a bow.—Gan. and Súr.

Again, $CH.\ CD = AC^2$ (Euclid III. 35)

$$\therefore AB = 2AC = 2\sqrt{CH.\ CD}.$$

$$\text{Also } \frac{AC^2}{CD} = CH;$$

$$\therefore DH = \frac{AC^2}{CD} + CD.]$$

208. Example. In a circle of which the diameter is ten, the chord being measured by six, say friend what the arrow is : and from the arrow tell the chord : and from chord and arrow, the diameter.

Statement : diameter 10. When the chord is 6, the length of the arrow comes out 1.

Or, the arrow being 1, the chord is found 6. Or from the chord and arrow the diameter is deduced 10.

209—211. Rule :[1] three stanzas. By 103923, 84853, 70534, 60000, 52055, 45922, and 41031, multiply the diameter of a circle, and divide the respective products by 120000; the quotients are severally, in their order, the sides of polygons from the triangle to the enneagon (inscribed) within the circle.[2]

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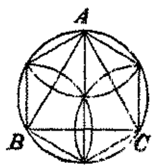
[1] To find the sides of regular inscribed polygons.—Gan. and Súr.

[2] Describe a circle with any radius at pleasure, divide the circumference into three equal parts and mark the points; and with these points (A, B, C) as centres and with the same radius, describe three circles, which will be equal in circumference to the first circle; and it is thus manifest that the side of the regular hexagon within the circle is half a diameter.



The side of an equilateral triangle (inscribed) in a circle is the upright, the diameter is hypotenuse and the side of the hexagon is side of a right-angled triangle. See above figure. Therefore the square root of the difference of the squares of the semi-diameter and diameter is the side of the (inscribed) equilateral triangle: *viz.*, for the proposed diameter (120000), 103923

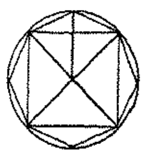
[In this rule the author gives the fractions by which the diameter of a circle is to be severally multiplied, in order to get the sides of inscribed regular figures from the triangle to the enneagon. Ganesa shows by a purely geometrical method how the fractions are arrived at, in the case of the triangle, the square, the hexagon, and the octagon; and remarks that a similar proof cannot be given in the case of the pentagon, the heptagon and the enneagon. See foot-note. Súryadása tries to supply the proofs in these cases, but his attempt is a failure; for the proofs he gives are not at all rigid and satisfactory, and it is not worth while to give them in the foot-note, as Colebrooke does. By the help of trigonometrical tables, proofs in all the cases may be given in a general manner as follows :—

Let a denote the side of a regular polygon of n sides inscribed in a circle of radius r. Then $a = 2r \sin\frac{\pi}{n}$ (see Todhunter's Trigonometry, Art. 255). Hence, side of inscribed equilateral triangle $= 2r \sin 60° = 2r\frac{\sqrt{3}}{2} = 2r\frac{1{\cdot}7320508\ldots}{2} = 2r \times {\cdot}8660254\ldots$ Now the fraction $\frac{103923}{120000} = {\cdot}866025$; thus the approximation is very close.

The side of the inscribed square $= 2r \sin\frac{\pi}{4} = 2r\frac{\sqrt{2}}{2} = 2r \times {\cdot}7071067\ldots.$

Now the fraction $\frac{84853}{120000} = {\cdot}707108\dot{3}$; thus the fraction is a little too large.

The side of a square is hypotenuse, and the semi-diameter is upright and side. Wherefore the square root of twice the square of the semi-diameter is the side of the (inscribed) square: *viz.*, for the diameter assumed, 84853.



The side of the regular octagon (see above figure) is hypotenuse, half the side of the square is upright, and the difference between that and the semi-diameter is the side. Wherefore the square root of the sum of the squares of half the side of the square and the semi-diameter less half the side of the square is the side of the (inscribed) regular octagon: *viz.*, for the diameter as put, 45922.

The proof of the sides of the regular pentagon, heptagon and enneagon cannot be shown in a similar manner.—Gan.

The side of the inscribed pentagon $= 2r \sin 36° = 2r \times \cdot 5877853$ from a table of natural sines. Now the fraction $\frac{70534}{120000} = \cdot 58778\dot{3}$; thus the fraction is a little too small.

The side of the inscribed hexagon is equal to the radius.

The side of the inscribed heptagon $= 2r \sin (25° \ 42' \ 51'')$ nearly $= 2r \times \cdot 4338819$ from the tables and the theory of proportional parts. Now the fraction $\frac{52055}{120000} = \cdot 433791\dot{6}$; thus the fraction is a little too small.

The side of the inscribed octagon $= 2r \sin 22\frac{1}{2}° = 2r \times \cdot 3826834$ from the tables. Now the fraction $\frac{45922}{120000} = \cdot 38268\dot{3}$; thus the approximation is very close.

The side of the inscribed nonagon $= 2r \sin 20° = 2r \times \cdot 3420201$ from the tables. Now the fraction $\frac{41031}{120000} = \cdot 341925$; thus the fraction is a little too small.

In the appendix to the *Goládhyáya*, called *Jyotpatti*, Bháskara has given an elaborate method of constructing the sines of various angles, adopting the old definition of the sine. (See Todhunter's Trigonometry, Art. 71.) The values deduced by his method closely approximate the values given in our modern tables, there being slight discrepancies in some cases, which account for the discrepancies noticed above between the values of the sides of some of the inscribed regular polygons as given in the text, and their values as calculated from the tables. A table of sines and versed sines of certain angles in arithmetical progression is also given in the *Súrya-siddhánta*, II. 15—27, the values there stated being less accurate than those deducible from Bháskara's method. See *Súrya-siddhánta*, Bápú Deva Sástrí's translation, II. 16, foot-note. The decimal notation is nowhere used either by Bháskara or in the *Súrya-siddhánta*. See note to § 138.

212. Example. Within a circle of which the diameter is two thousand, tell me severally the sides of the inscribed equilateral triangle and other polygons.

Statement: diameter 2000.

Answer: side of the triangle, $1732\frac{1}{20}$; of the tetragon, $1414\frac{13}{60}$; of the pentagon, $1175\frac{17}{30}$; of the hexagon, 1000; of the heptagon, $867\frac{7}{12}$; of the octagon, $765\frac{11}{30}$; of the nonagon, $683\frac{17}{20}$.

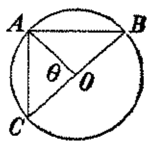
From variously assumed diameters, other chords are deducible, as will be shown by us under the head of construction of sines *(Jyotpatti)* in the treatise on Spherics.

[See *Goládhyáya*, appendix, Wilkinson's translation.]

The following rule teaches a short method of finding the gross chords.

213. Rule. The circumference less the arc being multiplied by the arc, the product is termed first.[1] From the quarter of the square of the circumference multiplied by five, subtract that first product, and by the remainder divide the first product taken into four times the diameter. The quotient will be the chord.

[This rule, as the author himself observes, gives a method of finding approximately the chords of given arcs of a circle. The commentators give an unsatisfactory and almost fanciful demonstration of the rule. The nature of the approximation may be shown thus:—



Let AB be the given arc whose chord is to be found. Draw the diameter BOC, and join AC, AO. Let θ denote the angle AOC, r the radius, and c the circumference of the circle. Then the value of the chord AB as given by the rule

$$= \frac{\text{arc } ACB \times \text{arc } AB \times 8r}{\frac{5}{4}c^2 - \text{arc } ACB \times \text{arc } AB}$$

$$= \frac{8r\left\{(\frac{1}{2}c)^2 - (\text{arc } CA)^2\right\}}{\frac{5}{4}c^2 - (\frac{1}{2}c)^2 + (\text{arc } CA)^2} = \frac{2r\,(4\pi^2 - 4\theta^2)}{4\pi^2 + \theta^2}$$

$$= 2r.\ \frac{1 - \left(\frac{\theta}{\pi}\right)^2}{1 + \left(\frac{\theta}{2\pi}\right)^2} = 2r.\left\{1 - \left(\frac{\theta}{\pi}\right)^2\right\}\left\{1 - \left(\frac{\theta}{2\pi}\right)^2 + \&c.\right\}$$

[1] *Prathama, ádya*, first (product).

$= 2r. \left(1 - \frac{5\theta^2}{4\pi^2}\right)$, neglecting powers of $\frac{\theta}{\pi}$ beyond the second. The accurate value of the chord $AB = 2r\cos\frac{\theta}{2} = 2r\left(1 - \frac{\theta^2}{8} + \&c.\right)$. Taking $\pi = \frac{22}{7}$, the value of the fraction $\frac{5}{4\pi^2}$ will be found to be very slightly greater than $\frac{1}{8}$. This shows the nature of the approximation. It is worthy of notice that the rule gives the exact value of the chord when the arc is a sixth part of the circumference. The rule appears to have been obtained empirically after repeated trials.]

214. Example. Where the semi-diameter is a hundred and twenty, and the arc of the circle is measured by an eighteenth multiplied by one and so forth (up to nine[1]), tell quickly the chords of those arcs.

Statement : diameter 240.

Here the circumference is 754 (nearly).

Arcs being taken which are multiples of an eighteenth of the circumference, the (corresponding) chords are to be sought.

Or for the sake of facility, abridging both circumference and arcs by the eighteenth part of the circumference, the same chords are found. Thus, circumference, 18 ; arcs, 1, 2, 3, 4, 5, 6, 7, 8, 9. Proceeding as directed, the chords come out 42, 82, 120, 154, 184, 208, 226, 236, 240.

In like manner, with other diameters (chords of assigned arcs may be found).[2]

[We can easily test the accuracy of the values of the chords given above by calculating their actual values from a table of

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[1] Up to nine, or half the number of arcs ; for the chords of the eighth and tenth will be the same, and so will those of the seventh and eleventh, and so forth.—Gan.

[2] Gang. &c.

natural sines. It will be found that the values given are in some cases less, and in others greater than the true values.]

215. Rule.[1] The square of the circumference is multiplied by a quarter of the chord and by five, and divided by the chord added to four times the diameter; the quotient being subtracted from a quarter of the square of the circumference, the square root of the remainder, taken from half the circumference, will leave the arc.

[This rule is derived from the preceding one. Denoting the arc AB (see figure in the note to §213) by x, we get $AB = \frac{8r(c-x)x}{\frac{5}{4}c^2-(c-x)x}$ (§ 213), whence $x^2 - cx + \frac{5\, AB.\, c^2}{4\,(8r+AB)} = 0$;

$\therefore \frac{c}{2} - x = \sqrt{\left\{\frac{c^2}{4} - \frac{5\, AB\, c^2}{4\,(8r+AB)}\right\}}$, the upper sign being taken, as x is supposed to be less than the semi-circumference. Hence the reason for the rule is obvious.

The following empirical and approximate rule for finding the arc is cited by Ganesa from Áryabhatta :—Six times the square of the arrow being added to the square of the chord, the square root of the sum is the arc. If 2θ denote the angle subtended by the arc at the centre, and r the radius, the expression for the arc as given by the rule

$= \sqrt{\{4r^2 \sin^2\theta + 6r^2 (1-\cos\theta)^2\}}$

$= r\sqrt{\{10 + 2\cos^2\theta - 12\cos\theta\}}$

$= r\sqrt{\left\{10 + 2\left(1 - \frac{\theta^2}{2} + \ldots\ldots\right)^2 - 12\left(1 - \frac{\theta^2}{2} + \ldots\ldots\right)\right\}}$

$= r\,.\,2\theta$, (neglecting higher powers of θ), which is the true value of the arc. This shows the nature of the approximation involved in the rule. In the case of the semi-circle, the rule gives for the length of the arc the expression $\sqrt{10}\, r$, so that the value of π is assumed to be $\sqrt{10}$.]

216. Example. From the chords which have been here found, now tell the length of the arcs, if, mathe-

[1] To find the arc from the chord given.

matician, thou have skill in computing the relation of arc and chord.

Statement: chords, 42, 82, 120, 154, 184, 208, 226, 236, 240.

Circumference abridged 18. The arcs thence found are, 1, 2, 3, 4, 5, 6, 7, 8, 9. They must be multiplied by the eighteenth part of the circumference.[1]

[Bháskara has given no rule for finding the area of a bow or segment of a circle. Ganesa cites in his commentary two rules for this purpose, which are practically the same. One of them, due to his father Kesava, is as follows:—The arrow being multiplied by half the sum of the chord and arrow, and a twentieth part of the product being added, the sum is the area of the segment. The other rule due to Srídhara is as follows:—The square of the arrow multiplied by the square of half the sum of the chord and arrow, being multiplied by ten and divided by nine, the square root of the product is the area of the bow. Since the fraction $\frac{21}{20}$ is very nearly equal to the fraction $\frac{\sqrt{10}}{3}$, we see that these two rules are practically the same. They both appear to be empirical and give very rough results, as may be readily seen by applying them to one or two particular cases. Thus, taking the first rule and applying it to the case when the segment is a semi-circle, we get for the area, the expression $\frac{21}{20}\,\frac{3r^2}{2}$, the true area being $\frac{\pi r^2}{2}$, r denoting the radius; so that in this case the value of π is taken to be $\frac{63}{20}$, which is greater than the true value.

[1] Súryadása and Gangádhara notice other figures omitted by the author, *e. g.*, *gaja-danta* or elephant's tusk, which may be treated as a triangle according to Srídhara; *bálendu* or crescent, which may be considered as composed of two triangles, according to the same author; *yava* or barleycorn, a convex lens, treated as consisting either of two triangles or two bows, according to Gangádhara; *nemi* or felloe; *vajra* or thunderbolt, treated as a quadrilateral with two bows, according to Gangádhara; *sankha* or conch; *mridanga* or great drum; and several others.

Again, applying the same rule to the case when the arc of the segment is a quadrant, we get for the area, the expression $\frac{21}{20} \cdot \frac{r^2}{4}$, the true area being $\frac{(\pi - 2) r^2}{4}$; so that in this case the value of π is taken to be $\frac{61}{20}$, which is less than the true value. Ganesa himself gives the accurate method of finding the area of the segment, namely, by subtracting the area of the triangle formed by the radii and the chord of the segment from the area of the sector. The same rule is also given in the *Manoranjana*.]

CHAPTER VII.

EXCAVATIONS[1] AND CONTENTS OF SOLIDS.

217—218. Rule[2]: a couplet and a half. Taking the breadth in several places,[3] let the sum of the measures be divided by the number of places: the quotient is the mean measure. So likewise with the length and depth.[4] The area of the plane figure, multiplied by the depth, will be the number of solid cubits contained in the excavation.

[The rule is very rough, giving a result much smaller than the true one. It is curious that such a rough rule was given when the author intended to lay down the correct rule immediately afterwards (§221). The tank contemplated is no doubt an ordinary one with slant sides, and we need not take the measurements in several places; the length and breadth of the mouth and bottom, and the depth of the bottom from the mouth, being sufficient for finding the volume accurately. See §221.]

219—220. Example: two stanzas. Where the length of the cavity, owing to the slant of the sides,

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[1] *Kháta-vyavahára.* The author treats first of excavations, secondly of stacks of bricks and the like, thirdly of sawing of timber, and fourthly of stores of grain, in as many distinct chapters.

[2] For measuring an excavation, the sides of which are trapezia.—Gan.

[3] *Vistára*, breadth; *dairghya*, length; *bedha*, depth. *Kháta*, an excavation; *sama-kháta*, a cavity in the form of a rectangular parallelopiped, cylinder, &c.; *vishama-kháta*, a cavity in the form of an irregular solid; *súchí-kháta*, an acute one, a pyramid or cone. *Sama-miti*, mean measure. *Ghana-phala*, the content of the excavation.

[4] The irregular solid is reduced to a regular one, to find its content.—Súr.

is measured by ten, eleven and twelve cubits in three several places, its breadth by six, five and seven, and its depth by two, four and three : tell me, friend, how many solid cubits are contained in that excavation.

Statement:	lengths,	12	11	10 ;
	breadths,	7	5	6 ;
	depths,	3	4	2.

Here finding the mean measure, the breadth is 6 cubits, the length 11, and the depth 3. The number of solid cubits is found, 198.

221. Rule[1] : a couplet and a half. The aggregate of the areas at the top and at the bottom, and of that resulting from the sum (of the sides of the summit and base), being divided by six, the quotient is the mean area : that multiplied by the depth is the neat[2] content.[3] A third part of the content of the regular equal solid is the content of the acute one.[4]

[This rule gives the exact volume. The tank contemplated is an ordinary one with uniformly slanting sides. Let *ABCD* be the mouth of the tank, and *EFGH* its base, both being supposed rectangular. Suppose the mouth of the tank to be covered by a plane. Draw perpendiculars on this plane from *E*, *F*, *G*, *H*, and from the feet of these perpendiculars, draw perpendiculars

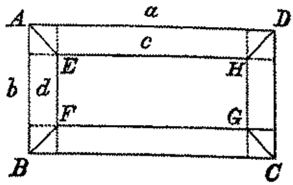
[1] To find the content of a prism, pyramid, cylinder and cone.

[2] Contrasted with the result of the preceding rule, which gave a gross or approximate measure.

[3] Half the sum of the breadths at the mouth and bottom is the mean breadth ; and half the sum of the lengths at the mouth and bottom is the mean length : their product is the area at the middle of the tank. (Four times that is the product of the sums of the length and breadth.) This, added to once the area at the mouth and once the area at the bottom, is six times the mean area.—Gan.

[4] As the bottom of the acute excavation is deep, by finding an area for it in the manner before directed, the regular equal solid is produced ; wherefore proportion is made : if such be th content, assuming three places, what is the content taking one? Thus the content of the regular equal solid, divided by three, is that of the acute one.—Súr.

on AB, BC, CD, DA, as in the figure. Let $AD=a, AB=b, EH=c, EF=d$, and $z=$ vertical depth of the tank. Then it will be easily seen that the tank is divided into a rectangular parallelopiped whose volume is cdz; four triangular prisms, two and two being equal, the united volume



being $\frac{1}{2}z\left\{(a-c)\ d+(b-d)c\right\}$; and four equal pyramids on square bases, one at each corner, the united volume being $\frac{1}{3}z\ (a-c)\ (b-d)$. Hence the volume of the tank

$=z\left\{cd+\frac{1}{2}(a-c)\ d+\frac{1}{2}(b-d)\ c+\frac{1}{3}(a-c)\ (b-d)\right\}$

$=z\times\frac{1}{6}\left\{ab+cd+(a+c)\ (b+d)\right\}$.

The last expression stated in words leads to the rule. The last part of the rule relating to the volumes of pyramids and cones is well known. Ganesa and Súryadása give curious demonstrations of the rule. See the foot-notes.]

222. Example. Tell the quantity of the excavation in a tank, of which the length and breadth are equal to twelve and ten cubits at its mouth, and half as much at the bottom, and of which the depth, friend, is seven cubits.

Statement : length 12; breadth 10; depth 7. The area at the mouth is 120 ; at the bottom 30 ; reckoned by the sum of the sides 270. Total 420. Mean area 70. Solid content 490.

223. Example. In a quadrangular excavation, the side being equal to twelve cubits, what is the content, if the depth be measured by nine? and in a round one, of which the diameter is ten and depth five ? and tell me separately, friend, the content of both acute solids.

Statement of quadrangular tank : side 12 ; depth 9. Proceeding as directed, the solid content comes out

1296. The content of the acute solid (quadrangular pyramid) is 432.

Statement of round tank: diameter 10; depth 5. The content nearly exact is $\frac{3927}{10}$; of the acute solid (cone), $\frac{1309}{10}$. Or gross content of the cylindrical tank is $\frac{2750}{7}$; of the cone, $\frac{2750}{21}$.

[The value of π is taken to be $\frac{3927}{1250}$. (See § 201).]

CHAPTER VIII.

STACKS.[1]

224—225. Rule[2]: a stanza and a half. The area of the plane figure (or base) of the stack,[3] multiplied by the height,[4] will be the solid content. The content of the whole pile, being divided by that of one brick, the number of bricks is found. The height of the stack, being divided by that of one brick, gives the number of layers.[5] So likewise with piles of stones.

[The stack is supposed to be in the form of a rectangular parallelopiped, and the reason for the rule is obvious. Bricks are, however, usually arranged in a pile so as to form a frustum of a quadrangular pyramid.]

226—227. Example: two stanzas. The bricks of a pile being eighteen fingers long, twelve broad and three high, and the stack being five cubits broad, eight long and three high, say what the solid content of the pile is; and what the number of bricks, and how many the layers.

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[1] *Chiti-vyavahára.*

[2] To find the solid content of a pile of bricks, or of stones or other things of uniform dimensions: also the number of bricks and of strata contained in the stack.

[3] *Chiti*, a pile or stack.

[4] *Uchchhráya*, *uchchhriti*, *auchchya*, height.

[5] *Stara*, layer or stratum.

Statement: length of pile, 8 ; breadth, 5 ; height, 3. Bricks, $\frac{3}{4}$ by $\frac{1}{2}$ by $\frac{1}{8}$.

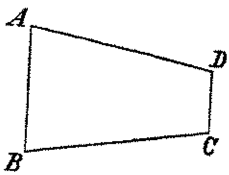
Answer. Solid content of the brick, $\frac{3}{64}$; of the stack, 120. Number of bricks, 2560. Number of layers, 24.

So likewise in the case of a pile of stones.

CHAPTER IX.

SAW.[1]

228. Rule: two half stanzas.[2] Half the sum of the thickness at both extremities, multiplied by the length in fingers, and the product again multiplied by the number of sections of the timber, and divided by five hundred and seventy-six,[3] will be the measure in cubits.



[The faces of the timber to which the sections are parallel, are supposed to be trapeziums, and the ends are supposed to be rectangular. Let $ABCD$ represent one of the sections. Then its area $= \frac{1}{2}(AB + CD) \times$ perpendicular distance between AB and $CD = \frac{1}{2}(AB + CD) \times AD$, nearly. The object of the reckoning is to settle the sawyer's charge which is at a certain rate for each square cubit along which the sawing is made. Hence the above area must be multiplied by the number of sections to get the total area for which the charge is to be reckoned.]

229. Example. Tell me quickly, friend, what the reckoning will be in cubits, for a timber the thickness

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[1] *Krakacha-vyavahára*, determination of the reckoning concerning the saw (*krakacha*) or iron instrument with a jagged edge for cutting wood.—Súr.

[2] The concluding half of one stanza begun in the preceding rule (§ 225), and the first half of another stanza of like metre completed in the following rule (§ 230).

[3] To reduce superficial fingers to superficial cubits.

of which is twenty fingers at the root, and sixteen fingers at the tip, and the length a hundred fingers, and which is cut by four sections.

Statement : length 100 ; thicknesses 20 and 16. Number of sections, 4.

Half the sum of the thicknesses at the two extremities, 18, multiplied by the length, makes 1800; this multiplied by the number of sections, gives 7200 ; divided by 576, gives the quotient in cubits, $\frac{25}{2}$.

230. Rule: half a stanza. But when the wood is cut across, the superficial measure is found by the multiplication of the thickness and breadth, in the mode above mentioned.[1]

[The reason for the rule is obvious.]

231. Maxim. The price for the stack of bricks or the pile of stones, or for excavation and sawing, is settled by the agreement of the workman, according to the softness or hardness of the materials.[2]

232. Example. Tell me what the superficial measure in cubits will be, for nine cross sections of a timber, of which the breadth is thirty-two fingers, and thickness sixteen.

Statement : breadth 32 ; thickness 16. Number of sections, 9.

Answer : 8 superficial cubits.

[The timber is supposed to be in the form of a rectangular parallelopiped.]

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[1] If the breadth be unequal, the mean breadth must be taken.—Gan. and Súr.

[2] This is levelled at certain preceding writers who have given rules for computing specific prices or wages, as A'rya-bhatta quoted by Ganesa, and as Brahmagupta (XII, 49); particularly in the instance of sawyer's work, by varying the divisors according to the difference of the timber.

CHAPTER X.

MOUND[1] OF GRAIN.

233. Rule. The tenth part of the circumference is equal to the depth (height[2]) in the case of coarse grain; the eleventh part, in that of fine; and the ninth, in the instance of bearded corn.[3] A sixth of the circumference being squared and multiplied by the depth (height), the product will be the solid cubits:[4] and they are *khárís* of *Magadha*.[5]

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[1] *Rási-vyavahára*, determination of a mound (of grain).

[2] *Bedha*, depth. Here it is the height in the middle from the ground to the summit of the mound.—Súr.

[3] *Anu*, *súkshma-dhánya*, fine grain, as mustard seed, &c.—Gan. As Paspalum Kora, &c.—*Mano.* As wheat, &c.—Súr.

Ananu, *sthúla-dhánya*, coarse grain, as chiches (cicer arietinum).—Gan. and Súr. As wheat, &c.—*Mano.* Barley, &c.—Chaturveda on Brahm.

Súkin, *súka-dhánya*, bearded corn, as rice, &c.

The coarser the grain, the higher the mound. The rule is founded on trial and experience; and for other sorts of grain, other proportions may be taken, as 9½ or 10½ or 12 times the height, equal to the circumference.—Gan. and Súr. The rule is taken from Brahmagupta, XII, 50.

[4] This is a rough calculation, in which the diameter is taken at one-third of the circumference. The content may be found with greater precision by taking a more nearly correct proportion between the circumference and diameter.—Gan.

[5] See § 7. The proportion of the *khárí* or other dry measure of any province to the solid cubit being determined, a rule may be readily formed for computing the number of such measures in a conical mound of grain. Ganesa accordingly delivers rules by him devised for the *khárí* of *Nandigráma* and for that of *Devagiri*: 'the circumference measured by the human cubit, squared and divided by sixteen, gives the *khárí* of *Nandigráma*; and by sixty, that of *Devagiri*.' (*Devagiri*, lit. mountain of the gods, is better

[The mound is supposed to be conical, the height being stated arbitrarily. The circumference of the base being given, the height will of course depend on the vertical angle of the cone. The rule is very rough, the value of π being taken equal to 3, as Ganesa remarks.

Let r denote the radius of the base, and h the height. Then the volume of the mound $=\frac{1}{3}\pi r^2 h=\frac{(2\pi r)^2}{3\times 4\pi}\times h=\left(\frac{\text{circumference}}{6}\right)^2 .h,$ supposing $\pi=3$.]

234. Example. Mathematician, tell me quickly how many *khárís* are contained in a mound of coarse grain standing on even ground, the circumference of which (mound) measures sixty cubits ; and separately say how many (are there) in a like mound of fine grain and in one of bearded corn.

Statement : circumference 60 ; height 6.

Answer : 600 *khárís* of coarse grain. But of fine grain, the height is $\frac{60}{11}$, and quantity thence deduced, $\frac{6000}{11}$. So, of bearded corn, the height is $\frac{60}{9}$, and quantity $\frac{6000}{9}$ *khárís.*

235. Rule. In the case of a mound piled against the side of a wall, or against the inside or outside of a corner of it, the product is to be sought with the circumference multiplied by two, four, and one and a third ; and is to be divided by its own multiplier.[1]

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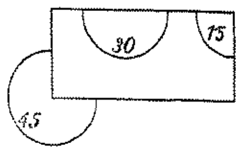
known by the name of *Daulatabad*, which the Emperor Muhammad conferred on it in the 14th century. *Nandigráma*, lit. the town or village of *Nandi*, Siva's bull and vehicle, retains the antique name, and is situated about 65 miles west of *Devagiri*.) He further observes that the cubit intended by the text is a measure in use with artisans, called in vulgar speech *gaj* ; and a *khárí* equal to such a solid cubit will contain twenty-five *manas* and three quarters.

[1] Against the wall, the mound is half a cone ; in the inner corner, a quarter of a cone ; and against the outer corner, three quarters. The circumference intended is a like portion of a circular base ; and the rule finds the content of a complete cone, and then divides it in the proportion of the part. See Gan.,&c.

[The reason for the rule is clear from the foot-note. The circumferences that are supposed to be given in the three cases are respectively half, one-fourth, and three-fourths of that of the base of the complete conical mound.]

236—237. Example : two stanzas. Tell me promptly, friend, the number of solid cubits contained in a mound of grain, which rests against the side of a wall, and the circumference of which measures thirty cubits ; and that contained in one piled in the inner corner and measuring fifteen cubits ; as also in one raised against the outer corner and measuring nine times five cubits.

Statement :



Twice the first mentioned circumference is 60. Four times the next is 60. The last multiplied by one and a third is likewise 60. With these the product is alike 600. This being divided by the respective multipliers, gives the several answers,[1] 300, 150 and 450.

[1] For coarse grain : but the product is $\frac{6000}{11}$ for fine, and $\frac{6000}{9}$ for bearded corn ; and the answers are $\frac{3000}{11}$, $\frac{1500}{11}$, $\frac{4500}{11}$; and $\frac{3000}{9}$, $\frac{1500}{9}$, $\frac{4500}{9}$.—Gan. &c.

CHAPTER XI.

SHADOW[1] OF A GNOMON.

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238. Rule.[2] The number five hundred and seventy-six being divided by the difference of the squares of the differences of both shadows and of the two hypotenuses,[3] and the quotient being added to one, the difference of the hypotenuses is multiplied by the square root of that sum ; and the product being added to, and subtracted from, the difference of the shadows, the moieties of the sum and difference are the shadows.

[The translation of the last sentence is not quite correct. It should be, " and the difference of the shadows being added to and subtracted from the product, the moieties, &c."

The rule, as the author hints in the example which follows (§239), is founded on the algebraic solution of a quadratic equation. Ganesa gives it at length after the manner of the author's *Víja-ganita*. It is however very long and not at all

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[1] *Chháyá-vyavahára*, determination of shadow; that is measurement by means of a gnomon.

[2] The difference of the shadows and difference of the hypotenuses being given, to find the length of the shadows and hypotenuses.—Súr.

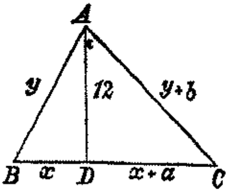
This rule is the first in the chapter, according to all the commentators except Súryadása, who begins with the next, §240, and places this after §244.

[3] *Chháyá*, *bhá*, *prabhá*, shadow.

Sanku, *nara*, *nri*, a gnomon, usually 12 fingers long.

Karna, hypotenuse of the triangle, of which the gnomon is the perpendicular, and the shadow the base.

clear, and so it has not been given in the foot-note. The rule may be demonstrated after the manner of modern algebra as follows :—



Let ABC be a triangle, and AD perpendicular to BC. Then BD, DC are called the shadows of the *sanku* or gnomon AD, which is supposed to be 12 fingers long ; BD, DC being supposed to be the shadows on a horizontal plane of the gnomon AD, produced either by sunlight at two different hours of the day, or by artificial light. The object of the rule is to find these shadows, the difference of AC, AB, and of CD, DB being supposed known.

Let $BD = x$, $DC = x + a$, $AB = y$, $AC = y + b$, a and b being known, and the measurements being in fingers.

Then $y^2 - x^2 = (y + b)^2 - (x + a)^2 = 144$;

$$\therefore\ by = ax + \frac{a^2 - b^2}{2},$$

whence by squaring and substituting $x^2 + 144$ for y^2, we obtain the quadratic

$$x^2 + ax + \left(\frac{a^2 - b^2}{4} - \frac{144b^2}{a^2 - b^2}\right) = 0,$$

solving which we get $x = \frac{1}{2}\left\{-a + b\sqrt{\left(1 + \frac{576}{a^2 - b^2}\right)}\right\}$,

(the upper sign only being admissible),

and $x + a = \frac{1}{2}\left\{a + b\sqrt{\left(1 + \frac{576}{a^2 - b^2}\right)}\right\}$.

These results stated in words lead to the rule. The rule is not of much importance.]

239. Example. The ingenious man, who tells the shadows of which the difference is measured by nineteen, and the difference of hypotenuses by thirteen, I take to be thoroughly acquainted with the whole of algebra as well as arithmetic.

Statement : difference of shadows, 19 ; difference of hypotenuses, 13. (Gnomon 12.)

Difference of their squares 192. By this divide 576 : quotient 3. Add one. Sum 4. Square root 2. By this multiply the difference of hypotenuses 13 : product 26. Add it to, and subtract it from, the difference of the shadows 19.

[The translation here is incorrect ; it should be, "add to it, and subtract from it, the difference, &c." See note to §238.]

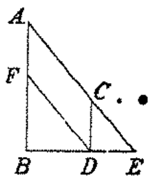
Half the sum and difference are the shadows, *viz.*, $\frac{45}{2}$ and $\frac{7}{2}$.

Under the rule in § 134, the gnomon being the upright, and the shadow the side, the square root of the sum of their squares is the hypotenuse. Thus the hypotenuses are $\frac{51}{2}$ and $\frac{25}{2}$.

240. Rule[1]: half a stanza. The gnomon multiplied by the distance of its foot from the foot of the light, and divided by the height of the torch's flame less the gnomon, will be the shadow.

[The rule follows from similar triangles as explained by Súryadása, as follows :—

Let A be the position of the light, CD the gnomon, and DE its shadow. From A draw AB perpendicular to ED produced. Through D draw DF parallel to AE. Then from the similar triangles CDE, FBD, we get $\frac{DE}{DC}=\frac{BD}{BF}$, whence $DE=\frac{BD \cdot DC}{AB-CD}$, $\because CD = AF$. Hence the rule.]



241. Example. If the base between the gnomon and torch be three cubits, and the elevation of the light, three cubits and a half, say quickly, friend, how

[1] The elevation of the light and (horizontal) distance of its foot from the foot of the gnomon being given, to find the shadow.— Gan.

much the shadow of a gnomon will be which measures twelve fingers.

Statement : gnomon $\frac{1}{2}$; distance between gnomon and torch, $\frac{7}{2}$; elevation of the light, 3.

Answer. Proceeding as directed, the shadow comes out 12 fingers.

242. Rule[1]: half a stanza. The gnomon being multiplied by the distance between it and the light, and divided by the shadow, and the quotient being added to the gnomon, the sum is the elevation of the torch.

[As Súryadása remarks, this rule also follows from similar triangles. See figure in the note to § 240.]

243. Example. If the base between the torch and gnomon be three cubits, and the shadow be equal to sixteen fingers, how much will be the elevation of the torch ? And tell me what the distance is between the torch and gnomon (if the elevation be given.)

Statement : distance between torch and gnomon, 3; shadow $\frac{2}{3}$.

Answer : height of the torch $\frac{11}{4}$.

244. Rule[2] : half a stanza. The shadow, multiplied by the elevation of the light less the gnomon and divided by the gnomon, will be the interval between the gnomon and light.

[This like the preceding rule also follows from similar triangles.]

Example, as before proposed (§ 243.)

Answer : distance 3 cubits.

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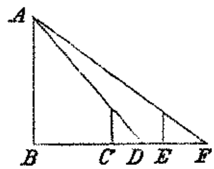
[1] To find the elevation of the torch, the length of the shadow, and the (horizontal) distance being given.—Súr.

[2] To find the (horizontal) distance, the elevation of the torch and length of the shadow being given.—Gan. and Súr.

245. **Rule**[1] : a stanza and a half. The length of a shadow multiplied by the distance between the terminations of the shadows and divided by the difference of the lengths of the shadows, will be the base. The product of the base and the gnomon, divided by the length of the shadow, gives the elevation of the torch's flame.[2]

In like manner is all this, which has been before declared, pervaded by the Rule of Three with its variations, as the universe is by the Deity.[3]



[Let A be the position of the light, C, E, the positions of the foot of the gnomon, and CD, EF, the corresponding shadows. Let $BC = x$, $BA = y$, $CD = a$, $EF = b$, $CE = c$, the measurements being in fingers.

Then from similar triangles we have,

$$\frac{y}{x+a} = \frac{12}{a}, \quad \frac{y}{x+b+c} = \frac{12}{b};$$

whence $x = \frac{ac}{b-a}$.

$$\therefore\ x + a = \frac{a(b+c-a)}{b-a} \ . \ . \ . \ . \ . \ (1),$$

$$\text{and } y = \frac{12\ (x+a)}{a} = \frac{12\ (x+b+c)}{b},$$

whence the reason for the rule is obvious. Bháskara's own explanation is practically the same as the above, but it is not clearly put. He at once states a proportion which is equivalent to equation (1) above, but he does not explain how it is obtained.]

[1] The gnomon being set up successively in two places, the distance between which is known, and the length of the two shadows being given, to find the elevation of the light, and the base.—Gan and Súr.

[2] The rule is borrowed from Brahmagupta (XII, 54).

[3] The author intimates that the whole preceding system of computation, as well as the rules contained under the present head, as those before delivered, is founded on the rule of proportion.—Gan.

246. Example. The shadow of a gnomon measuring twelve fingers being found to be eight, and that of the same placed on a spot two cubits further in the same direction, being measured twelve fingers, say, intelligent mathematician, how much the distance of the shadow[1] from the torch is, and the height of the light, if thou be conversant with computation, as it is termed, of shadow.

Statement: shadows, 8, 12; interval between the positions of the foot of gnomon, 48.

Here the interval between the termination of the shadows is in fingers 52. The first shadow 8, multiplied by the interval 52, and divided by the difference of the length of the shadows, *viz.*, 4, gives the length of the base 104. It is the distance between the foot of the torch and the tip of the first shadow. So the length of the base to the tip of the second shadow is 156.

The product of the base and gnomon, divided by the shadow, gives both ways the same elevation of the light, *viz.*, 6½ cubits.

"In like manner, &c."[2]—Under the present head of measurement of shadow, the solution is obtained by putting a proportion: *viz.*, if so much of the shadow, as is the excess of the second above the first, give the base intercepted between the tips of the shadows, what will the first give? The distances of the terminations of the shadows from the foot of the torch are in this manner severally found. Then a second pro-

[1] All the commentators appear to have read 'gnomon' in this place; but one copy of the text exhibits 'shadow' as the reading: and this seems to be correct.

[2] Reference to the text, §245. [The author here purports to explain fully what he has hinted at before.—Ed.]

portion is put: if, the shadow being the side, the gnomon be the upright; then, the base being the side, what will be the upright? The elevation of the torch is thus found; and is both ways (that is, computed with either shadow,) alike.

[See note to § 245.]

So the whole sets of five or more terms are explained by twice putting three terms and so forth.

As the Being, who relieves the minds of his worshippers from suffering, and who is the sole cause of the production of this universe, pervades the whole, and does so with his various manifestations, as worlds, paradises,[1] mountains, rivers, gods, demons, men, trees,[2] and cities; so is all this collection of instructions for computations pervaded by the rule of three terms. Then why has it been set forth by so many different (writers[3], with much labour and at great length)? The answer is:—

247. Whatever is computed either in algebra or in this (arithmetic) by means of a multiplier and a divisor, may be comprehended by the sagacious learned as the rule of three terms. Yet has it been composed by wise instructors in miscellaneous and other manifold rules, teaching its easy variations, thinking thereby to increase the intelligence of such dull comprehensions as ours.

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[1] *Bhuvana*, worlds. *Bhavana*, paradises, the abodes of Brahmá and the rest of the gods. [The reading here adopted by Colebrooke is apparently different from that in Pandit Jívánanda Vidyáságara's edition in which we have *sakala-bhuvana-bhávanena*, which rendered becomes, 'who is the creator of all the worlds,' the words, 'worlds' and 'paradises' in Colebrooke's translation, being omitted in that case.—Ed.]

[2] *Naga*, either tree or mountain. The term, however, is read in the text by none of the commentators besides Ganesa.

[3] As Srídhara and the rest.—*Mano.* As Brahmagupta and others.—Gang.

CHAPTER XII.

PULVERIZER.

248—252. Rule: five stanzas.

248. In the first place, as preparatory to the investigation of a pulverizer,[1] the dividend, divisor and

[1] *Kuttaka-vyavahára* or *kuttakádhyáya*, determination of a grinding or pulverizing multiplier, or quantity such that a given number being multiplied by it, and the product added to a given quantity, the sum (or, if the additive be negative, the difference) may be divisible by a given divisor without remainder. *Kuttaka* or *kutta* from *kutt*, to grind or pulverize; (to multiply: all verbs importing tendency to destruction also signifying multiplication.—Gan.) The derivative import of the word is retained in the present version to distinguish it from multiplier in general; *kuttaka* being restricted to the particular multiplier of the problem in question.

According to the remark of Ganesa, this chapter as well as the following chapter on combination belongs to algebra rather than arithmetic; and they are here introduced, as he observes, and treated without employing algebraic forms, to gratify such as are unacquainted with analysis. See *Vija-ganita*, Chap. II, from which the present chapter is borrowed, the contents being copied, with some variation of the order, nearly word for word.

Ganesa notices an objection, namely, that this subject ought not to have been introduced into a treatise on arithmetic, while a passage of A'ryabhatta expressly distinguishes it from both arithmetic and algebra: "the multifarious doctrine of the planets, arithmetic, the pulverizer (*kuttaka*), and analysis (*vija*) and the rest of the science treating of seen (or physical) objects." He answers the objection by saying that mathematics (*ganita*) consists of two branches treating of known and of unknown quantity (*vyakta-ganita* and *avyakta-ganita*); that the investigation of the pulverizer is comprehended in algebra; and that the separate mention of this subject by A'ryabhatta and other ancient authors is intended to indicate its difficulty and importance. In Brahmagupta's work, the whole of algebra is comprised under the title of *kuttakádhyáya*, chapter on the pulverizer. See Brahm., Ch. XVIII.

additive quantity[1] are, if practicable, to be reduced by some number.[2] If the number, by which the dividend and divisor are both measured, do not also measure the additive quantity, the question is an ill put (or impossible) one.

249—251. The last remainder, when the dividend and divisor are mutually divided, is their common measure.[2] Being divided by that common measure, they are termed reduced quantities.[3] Divide mutually the reduced dividend and divisor, until unity be the remainder in the dividend. Place the quotients one under the other, and the additive quantity beneath them, and cipher at the bottom. By the penult multiply the number next above it and add the lowest term. Then reject the last and repeat the operation until a pair of numbers be left. The uppermost of these being abraded[4] by the reduced dividend, the remainder is the quotient. The other (or lowermost) being in like manner abraded by the reduced divisor, the remainder is the multiplier.

252. * Thus precisely is the operation when the number of quotients is even. But if the number be

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[1] *Kshepa* or *yuti*, additive ; from *kship* to cast or throw in, and from *yu* to mix. *Visuddhi*, subtractive quantity.

[2] *Apavartana*, abridgment.—Gan. Reduction to least terms, division without remainder ; also the number which serves to divide without residue, the common measure.—Súr. [It is really the greatest common measure.—Ed.]

[3] *Dridha*, firm ; reduced by the common divisor to the least term. The word is applicable to the reduced additive, as well as to the dividend and divisor.

[4] *Tashta* abraded ; from *taksh*, to pare or abrade : divided, but the residue taken, disregarding the quotient.—Súr. As it were a residue after repeated subtractions.—Gang.

Takshana, the abrader ; the diviser employed in such operation.

odd, the numbers as found must be subtracted from their respective abraders, and the residues will be the true quotient and multiplier.

[This Chapter, as Ganesa remarks, properly belongs to Algebra and not to Arithmetic. We have already seen, however, that the present treatise deals with both Arithmetic and Algebra.

The whole of this Chapter is occupied with problems producing indeterminate equations of the first degree, and the object of the several rules is to find positive integral solutions of such equations. The reason for the above rule will be best understood from the example in § 253. Let y denote the multiplier and x the integral quotient. Then we get $\frac{221x + 65}{195} = x$, and the object of the rule is to find positive integral solutions of this equation.

Dividing by the common measure 13, we get $\frac{17y + 5}{15} = x$,

$\therefore\ 15x - 17y = 5 \ldots\ldots (1)$.

This equation is of the type $Ax - By = C$, A being less than B, and the rule refers to cases of this class. Convert $\frac{B}{A}$ into a continued fraction, and suppose the result is $\frac{B}{A} = a + \frac{1}{b+} \frac{1}{c + \&c.}$. Let $\frac{q}{p}$ be the convergent immediately preceding $\frac{B}{A}$. Then we know that $x = qC$, $y = pC$, or $x = (B - q)\ C$, $y = (A - p)\ C$, is one solution of equation (1), according as $Aq - Bp = \pm 1$; and that the general solution is $x = \alpha + Bt$, $y = \beta + At$, where α and β are one set of values of x and y, and where we may give to t any positive integral value, and also such negative integral values as make Bt and At numerically less than α and β respectively. (See Todhunter's Algebra, Arts. 630, 631.) Now the succesive convergents to $\frac{B}{A}$ are $\frac{a}{1}$, $\frac{ab+1}{b}$, $\frac{abc+a+c}{bc+1}$, &c., the law of formation being well known (see Todhunter's Algebra, Art. 604); and the object of the rule in § 251 is to find the value of the convergent $\frac{q}{p}$, and thence the value of the quantities qC, pC.

The rule, however, is very vaguely and obscurely expressed, and it is difficult to understand its working. An explanation of the rule by means of a particular example is given by Krishna in his commentary on the *Víja-ganita*, from which we may deduce the following general explanation. Let us first consider the case where the number of quotients exclusive of the last one is two, *viz.*, a, b, the additive being C. Then according to the rule we get the series a, b, C, 0. The rule next directs us to multiply the penultimate (C) by the number preceding it, *viz.*, b, and to add the last term, *viz.*, 0, to the product. We thus get $bC+0$ or bC, and we have now to replace the previous multiplier b by this quantity bC, and reject the last term 0. We thus have the new series, a, bC, C, with which we have to repeat the above operation: that is, we have to multiply the penultimate bC, by a, and add the last term C to the product; whence we get $abC+C$, by which we have to replace the multiplier a, and we have to reject C. The series thus becomes $abC+C$, bC, and this consisting only of two terms, we infer according to the rule that $qC=abC+C$, and $pC=bC$. But we know as a matter of fact that in the case we are considering, $q=ab+1$, and $p=b$; thus the rule holds good in this case. We may now prove by induction that the rule holds universally. For supposing the number of quotients exclusive of the last one to be three, *viz.*, a, b, c, all we have got to do is to take the additional quotient c, so that we have to write $b+\frac{1}{c}$ for b in the above expressions for qC, pC; and we get

in this case $\dfrac{qC}{pC}=\dfrac{a\left(b+\frac{1}{c}\right)C+C}{\left(b+\frac{1}{c}\right)C}=\dfrac{a(bcC+C)+cC}{bcC+C}$, so that

$qC=a\ (bcC+C)+cC$, and $pC=bcC+C$. But it is easy to see from the very nature of these expressions that they are precisely what we would get by means of the rule, if we take into consideration the quotient c. Thus we see that the rule is universally true. In the present example, $a=1$, $b=7$, $c=2$, and 7 is the last quotient but one. Hence, $q=ab+1=7+1$, and $qC=7\times5+5$; $p=b=7$, and $pC=7\times5$. And $\because$ in this case $Aq-Bp$

$=1$, $\therefore x=40$, $y=35$, is one solution. Hence putting $\alpha=40$, $\beta=35$, and $t=-2$, in the general expressions for x and y, we get $x=40-17\times 2=6$, $y=35-15\times 2=5$. These are the least positive integral values. Putting $t=-1$, we get $x=23$, $y=20$, and so on. Or taking $x=6$, $y=5$, as one solution, we may get others from the expressions $6+17t$, $5+15t$, by giving to t any positive integral value. Thus putting $t=1$, we get $x=23$, $y=20$; putting $t=2$, we get $x=40$, $y=35$; and so on. Thus the reason for the process in § 253, as well as that for the rule in § 262, is clear. The meaning of the term *abraded* as used by the author is also clear.]

253. Example. Say quickly, mathematician, what that multiplier is, by which two hundred and twenty-one being multiplied, and sixty-five added to the product, the sum divided by a hundred and ninety-five becomes exhausted.

Statement: dividend 221. divisor 195. Addit ve 65.

Here the dividend and divisor being mutually divided, the last of the remainders (or divisors) is 13. By this common measure, the dividend, divisor and additive, being reduced to their least terms, are dividend 17, divisor 15, additive 5. The reduced dividend and divisor being divided reciprocally, and the quotients put one under the other, the additive under them, and cipher at the bottom, the series[1] which results is 1.
7
5
0

Then multiplying by the penult the number above it and proceeding as directed, the two quantities are obtained 40. These being abraded by the reduced
35

[1] *Valli*, series.

dividend and divisor 17 and 15, the quotient and multiplier are obtained 6 and 5. Or, by the subsequent rule (§ 262), adding them to their abraders multiplied by an assumed number, the quotient and multiplier (putting 1) are 23 and 20 ; or putting 2, they are 40 and 35; and so forth.

254. Rule. The multiplier is also found by the method of the pulverizer, the additive quantity and dividend, being either reduced by a common measure (or used unreduced[1]). But if the additive and divisor be so reduced, the multiplier found, being multiplied by the common measure, is the true one.

[The reason for the rule will appear from the solution of the example in § 255.]

255. Example. If thou be expert in the investigation of such questions, tell me the precise multiplier by which a hundred being multiplied, with ninety added to the product, or subtracted from it,[2] the sum or the difference may be divisible by sixty-three without a remainder.

Statement: dividend 100. divisor 63. Additive 90.

The quotient and multiplier are found by proceeding as before, 30 and 18.

Or, the dividend and additive being reduced by the common measure ten, we get dividend 10, divisor 63, additive 9. Placing the quotients of reciprocal division, the additive quantity and cipher, one under the other,

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[1] Gan.

[2] An example of the subsequent rule in § 256.

the series is 0 6 3 9 0 And the multiplier is found by the former process 45. The quotient[1] (3) is here not to be taken; and the number of quotients (of the series) being odd, the multiplier 45 is to be subtracted from its own abrader 63; the true multiplier is thus found 18. The dividend being multiplied by that multiplier, and the additive quantity being added, and the sum divided by the divisor, the quotient is found 30.

Or, the divisor and additive quantity being reduced by the common measure nine, we get dividend 100, divisor 7, additive 10. Here the quotients, the additive and cipher make the series 14. 3 10 0 The multiplier is found 2, which multiplied by the common measure 9, gives the true multiplier 18.

Or, the dividend and additive being reduced, and further the divisor and additive, by common measures, we get dividend 10, divisor 7, additive 1. Proceeding as before the series is 1. 2 1 0 The multiplier hence deduced is 2, which taken into the common measure 9, gives 18; and hence, by multiplication and division, the quotient comes out 30.

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[1] [This probably means the *last* quotient. There is, however, no force in the remark; the last quotient being always excluded under the rule in §§ 250—251.—Ed.]

Or, adding the quotient and multiplier as found, to (multiples of) their respective divisors, the quotient and multiplier are 130 and 81 ; or 230 and 144 ; and so forth.

[Putting $y=$ multiplier, and $x=$ quotient, we get the equation $63x-100y=90$......(1). If we convert $\frac{100}{63}$ into a continued fraction, the number of quotients will be found to be large, and so it will be tedious to form $\frac{q}{p}$. To make the process shorter, let us first consider the equation $63x-10y=9$...... (2). Now if from (2) we find sets of positive integral values of x and y, it is clear that those values of y and 10 times the corresponding values of x will be sets of positive integral values of x and y satisfying (1). The general solution of (2) will be found to be $x=(10-3)9+10t$, $y=(63-19)9+63t$. Putting $t=-6$, we get $x=3$, $y=18$, as the least values. Thus $x=30$, $y=18$, are the least values satisfying (1). In the text, the least value of y is found from the expression $63-(19\times 9+63t')$, by putting $t'=-2$, and the reason for this is stated to be the fact that the number of quotients (exclusive of the last one) is odd. The explanation is that the number of quotients exclusive of the last one being odd, $\frac{q}{p}$ is less than $\frac{B}{A}$ (Todhunter's Algebra, Art. 603), and so $Aq-Bp=-1$, and $\therefore$ the general solution is $x=(B-q)C+Bt$, $y=(A-p)C+At$......(3). In the text, the value of y is derived from the expression $A-(pC+At')$...(4). Supposing $\therefore$ that the expressions in (3) and (4) give the same values of y for certain values of t and t', the relation between such values will be given by $t+t'=1-C$. Thus it is clear that we can derive values of y from both these expressions, but not from the expression $pC+At$, when the number of quotients exclusive of the last one is odd. This also explains the statement in §252.

Similarly, if we find values of x and y from the equation $7x-100y=10$, these values of x and 9 times the corresponding values of y will satisfy (1).

Lastly, if we find values of x and y from the equation $7x-10y=1$, it is easy to see that 10 times these values of x, and 9 times the corresponding values of y will satisfy (1). Thus the reason for the rule in § 254 is obvious.]

256. Rule[1] : half a stanza. The multiplier and quotient, as found for an additive quantity, being subtracted from their respective abraders, answer for the same as a subtractive quantity.

Here the quotient and multiplier as found for the additive quantity ninety in the preceding example, namely, 30 and 18, being subtracted from their respective abraders, namely, 100 and 63, the remainders are the quotient and multiplier which answer when ninety is subtractive : *viz.*, 70 and 45.

Or, these being added to arbitrary multiples of their respective abraders, the quotient and multiplier are 170 and 108, or 270 and 171, &c.

[Let C be the additive or subtractive quantity. Then the corresponding equations will be $Ax-By=\ \ C$......(1).

$$Ax-By=-C\text{......(2)}.$$

Let $x=\alpha$, $y=\beta$, be a solution of (1). Then $A\alpha-B\ \ =C$.

$\therefore A(B-\alpha)-B(A-\beta)=-C$.

$\therefore x=B-\alpha$, $y=A-\beta$, is a solution of (2), whence the reason for the rule is clear. It will be readily seen that the general solution of (2) is $x=(B-\alpha)+Bt$, $y=(A-\beta)+At$.]

257. Another example[2]. Tell me, mathematician, the multipliers severally, by which sixty being multiplied, and sixteen being added to the product, or subtracted from it, the sum or difference may be divisible by thirteen without a remainder.

[1] The rule serves when the additive quantity is negative.—Gan. and Súr.

[2] This additional example is unnoticed by Ganesa, but expounded by the rest of the commentators, and found in all copies of the text that have been collated.

Statement : dividend 60, divisor 13. Additive 16.

The series found as before, is 4.
1
1
1
1
16
0

Hence the multiplier and quotient are deduced 2 and 8. But the number of quotients (of the series) is here uneven ; wherefore the multiplier and quotient must be subtracted from their abraders 13 and 60 ; and the multiplier and quotient, answering to the additive quantity sixteen, are 11 and 52. These being subtracted from the abraders, the multiplier and quotient, corresponding to the subtractive quantity sixteen, are 2 and 8.

258. Rule[1] : a stanza and a half. The intelligent calculator should take a like quotient (of both divisions) in the abrading of the numbers for the multiplier and quotient (sought). But the multiplier and quotient may be found as before, the additive quantity being (first) abraded by the divisor ; the quotient, however, must have added to it the quotient obtained in the abrading of the additive. But in the case of a subtractive quantity, it is subtracted.

[The reason for the rule will appear from the solution of the example which follows.]

259. Example. What is the multiplier, by which five being multiplied and twenty-three added to the

[1] Applicable when the additive quantity exceeds the dividend and divisor. —Gan.

product, or subtracted from it, the sum or difference may be divided by three without remainder?

Statement: dividend 5 / divisor 3. Additive 23.

Here the series is 1 and the pair of numbers found
1
23
0
as before 46 / 23. They are abraded by the dividend and divisor, respectively. The lower number being abraded by 3, the quotient is 7 (and residue 2). The upper number being abraded by 5, the quotient would be 9 (and residue 1); nine, however, is not to be taken; but, under the rule for taking like quotients, seven only, (and the residue consequently is 11). Thus the multiplier and quotient come out 2 and 11.

And by the former rule (§ 256) the multiplier and quotient answering to the same as a negative quantity are found, 1 and the negative quantity—6. Added to arbitrary multiples of their abraders, double for example, so that the quotient may be positive, the multiplier and quotient are 7 and 4. So in every (similar) case.

Or, statement for the second (part of the) rule: dividend 5 / divisor 3. Additive abraded[1] 2.

The multiplier and quotient hence found as before are 2 and 4. These subtracted from their respective divisors, give 1 and 1, as answering to the subtractive quantity. The quotient obtained in the abrading of the additive, (*viz.* 7) being added in one instance and

[1] 23, abraded by the divisor 3, gives the quotient 7 and residue 2.

subtracted in the other, the results are 2 and 11 answering to the additive quantity, and 1 and -6 answering to the subtractive: or, to obtain a positive quotient, add to the latter twice their divisors, and the result is 7 and 4.

[Putting y = multiplier, and x = quotient, we get the equations $3x-5y=\pm 23$...(1). Taking the upper sign, the general solution will be found to be $x=2\times 23+5t$, $y=1\times 23+3t$. The least positive integral values are got by putting $t=-7$, *viz.*, $x=11$, $y=2$. The meaning of the first part of the rule in §258 is that the *same negative* value is to be given to t in the expressions for x and y. To explain the second part of the rule, we observe that equation (1) may be written $\frac{5y+2}{3}=x\mp 7=X$ suppose. We may then solve $3X-5y=2$......(2), the values of y being the same in (1) and (2), and the values of x being deducible from those of X, from the relation $x=X\pm 7$.]

260. Rule[1]: one stanza. If there be no additive quantity, or if the additive be measured by the divisor, the multiplier may be considered as cipher, and the quotient as the additive divided by the divisor.[2]

[The rule is clear enough.]

261. Example. Tell me promptly, mathematician, the multiplier by which five being multiplied and added to cipher, or added to sixty-five, the division by thirteen shall in both cases be without remainder.

Statement: dividend 5, divisor 13. Additive 0.

There being no additive, the multiplier and quotient are 0 and 0; or 13 and 5; or 26 and 10; and so forth.

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[1] Applicable if there be no additive, or if it be divisible by the divisor without remainder.

[2] It is so in the latter case; but in the former (where the additive is null), the quotient is cipher.—Súr.

Statement: dividend 5 / divisor 13 . Additive 65.

By the rule (§ 260), the multiplier and quotient come out 0 and 5; or 13 and 10; or 26 and 15; and so forth.

[Putting y = multiplier, and x = quotient, we get in the first case the equation $\frac{5y+0}{13}=x$; and in the second case, the equation $\frac{5y+65}{13}=x$. The general solution in positive integers of the first equation will be readily found to be $x=5r$, $y=13r$; and that of the second, $x=5(1+r)$, $y=13r$, where r may be zero or any positive integer.]

Rule.[1] Or, the dividend and additive being abraded by the divisor, the multiplier may thence be found as before; and the quotient from it, by multiplying the dividend, adding the additive, and dividing by the divisor.

In the former example (§ 253), the reduced dividend, divisor and additive respectively are, 17, 15, 5. Abraded by the divisor (15) the additive and dividend become 5 and 2; and the statement is:—

dividend 2 / divisor 15 . Additive 5.

Proceeding as before the two terms found are 5, 35. The second one, abraded by the divisor (15), gives the multiplier 5; whence, by multiplying with it the dividend (17) and adding (the additive), and dividing (by the divisor), the quotient comes out 6.

[The reason for the above rule is clear. Let the equation be $Ax-By=C$, and suppose B greater than A, and C less than

[1] This is found in one copy of the text, and is expounded only by Gangádhara, being unnoticed by the other commentators. It occurs, however, in the similar chapter of the *Vija-ganita*, §62.

A. Divide B by A; let K denote the quotient, and B' the remainder. Thus $B=KA+B'$, and $\therefore \frac{B}{A}=K+\frac{B'}{A}$. Convert $\frac{B}{A}$ and $\frac{B'}{A}$ into continued fractions, and let $\frac{q}{p}$ and $\frac{q'}{p'}$ be the convergents immediately preceding $\frac{B}{A}$ and $\frac{B'}{A}$. Then $\frac{q}{p}=K+\frac{q'}{p'}=\frac{Kp'+q'}{p}$. Thus $p=p'$, and $Aq-Bp=A(Kp'+q')-(KA+B')p'$ $=Aq'-B'p'$. Hence it is evident that the values of y found from $Ax-B'y=C$, will be the same as those found from $Ax-By$ $=C$; and y being known, x is of course known from the equation $x=\frac{By+C}{A}$. It must be remarked here that the above rule applies only when the additive C is less than the divisor A, so that the additive abraded by the divisor remains unchanged.]

262. Rule for finding divers multipliers and quotients in every case : half a stanza. The multiplier and quotient, being added to their respective (abrading) divisors multiplied by assumed numbers, become manifold.

The influence and operation of this rule have been already shown in various instances.

[See note to § 252.]

263. Rule for a constant pulverizer[1] : one stanza. Unity being taken for the additive quantity, or for the subtractive, the multiplier and quotient, which may be thence deduced, being severally multiplied by an arbitrary additive or subtractive, and abraded by the respective divisors, will be the multiplier and quotient for such assumed quantity.

In the first example (§253), the reduced dividend and divisor with additive unity furnish this state-

[1] *Sthira-kuttaka*, steady pulverizer.

ment : dividend 17 / divisor 15 . Additive 1.

Here the multiplier and quotient (found in the usual manner) are 7 and 8. These multiplied by an assumed additive five, and abraded by the respective divisors 15 and 17, give the multiplier and quotient 5 and 6, for that additive.

Next, unity being the subtractive quantity, the multiplier and quotient thence deduced are 8 and 9. These multiplied by five and abraded by the respective divisors, give 10 and 11.

So in every (similar) case.

Of this method of investigation great use is made in the computation of planets. On that account something is here said (by way of instance.)

[The above rule is not a new one. The equation is supposed to be $Ax-By=\pm 1$...(1), and from what we have already seen, the general solution of this is $x=q+Bt$, or $=(B-q)+Bt$, and $y=p+At$, or $=(A-p)+At$. If now the additive or subtractive be any integer whatever, *i. e.*, if the equation be $Ax-By=\pm C$...(2), we have only to multiply q or $B-q$, and p or $A-p$ by C in the above expressions for x and y, in order to get the general solution of equation (2). We may thus regard the general value of y found from (1) as a *steady* quantity from which we may derive the general value of y satisfying (2). This shows the propriety of the expression *constant pulverizer.*]

264. A stanza and a half. Let the remainder of seconds be made the subtractive quantity,[1] sixty the dividend, and terrestrial days[2] the divisor. The quo-

[1] The present rule is for finding a planet's place and the elapsed time, when the fraction above seconds is alone given.—Gan.

[2] The number of terrestrial days in a *kalpa* is stated at 1577916450000. See the *Ganitádhyáya* of the *Siddhánta-siromani*, I, 20—21. [By a *terrestrial* day is meant the *mean solar* day, when it is taken for the purpose of

tient deduced therefrom will be the seconds ; and the multiplier will be the remainder of minutes. From this again the minutes and remainder of degrees are found; and so on upwards. In like manner, from the remainder of exceeding months and deficient days,[1] may be found the solar and lunar days.

The finding of (the place of) the planet and the elapsed days, from the remainder of seconds in the planet's place, is thus shown. Sixty is there made the dividend ; terrestrial days, the divisor ; and the remainder of seconds, the subtractive quantity : with which the multiplier and quotient are to be found. The quotient will be seconds ; and the multiplier, the remainder of minutes. From this remainder of minutes taken (as the subtractive quantity), the quotient deduced will be minutes ; and the multiplier, the remainder of degrees. The residue of degrees is next the subtractive quantity; terrestrial days, the divisor ; and thirty, the dividend : the quotient will be degrees ; and the multiplier, the remainder of signs. Then twelve is made the dividend; terrestrial days, the divisor ; and the remainder of signs the subtractive quantity : the quotient will be signs ; and the multiplier, the remainder of revolutions. Lastly, the revolutions in a *kalpa* become the dividend ; terrestrial days, the divisor ; and the remainder of re-

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astronomical measurement ; but for practical purposes, it is taken as the time from sunrise to sunrise, which would make its duration variable. See *Goládhyáya*, Wilkinson's translation, II, 3, Bápú Deva Sástrí's note : *Súrya-siddhánta*, Burgess's translation, I, 34—40, note.—Ed.]

[1] *Adhi-mása*, additive months ; *Avamadina*, subtractive days. See *Ganitádhyáya*, I, 42. [See also *Goládhyáya*, IV, 10—16, note : *Súrya-siddhánta*, I, 47—50, note.—Ed.] The *adhimásas* in a *kalpa* are 1593300000 [$=1602999000000 \div 30 - 4320000000 \times 12$], being the excess of the lunar over the solar months. The *avamas* in a *kalpa* are 25082550000, being the excess of the lunar days over the terrestrial days.

volutions, the subtractive quantity : the quotient will be the elapsed revolutions ; and the multiplier, the number of elapsed days.[1] Examples of this occur (in the *Siromani*) in the chapter of the problems[2] (*Triprasnádhyáya*).

In like manner, the exceeding months in a *kalpa* are made the dividend ; solar days,[3] the divisor ; and the remainder of exceeding months, the subtractive quantity : the quotient will be the elapsed additional months; and the multiplier, the elapsed solar days. So the deficient days in a *kalpa* are made the dividend ; lunar days,[4] the divisor ; and the remainder of deficient days, the subtractive quantity : the quotient will be the elapsed fewer days ; and the multiplier the elapsed lunar days.

[The reason for the rule for finding a planet's place and the elapsed time will be best understood from the illustration given by Ganesa and Gangádhara in arbitrary numbers. Put the terrestrial days in a *kalpa* 19, the revolutions of the planet in the *kalpa* 10, the elapsed days 12. Then we evidently get the proportion, 19 : 12 :: 10 : number of revolutions already performed by the planet, whence the revolutions = $6\frac{6}{19}$. Thus the planet has performed 6 complete revolutions, and $\frac{6}{19}$ of a revolution, so that to find the planet's place, we must reduce the

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[1] The elapsed days of the *kalpa* to the time for which the planet's place is found. See *Ganitádhyáyá*, I, 47—49.

[2] [See also *Goládhyáya*, Chap. XIII.—Ed.]

[3] The solar days in a *kalpa* are 1555200000000 [=4320000000×360]. See *Ganitádhyáya*, I, 40.

[The number of solar years in a *kalpa* is 4320000000. See *Súrya-siddhánta*, I, 19, note.—Ed.]

[4] The lunar days, reckoning thirty to the month or synodical revolution, are 1602999000000 in the *kalpa*. See *Ganitádhyáya*, I, 40. [See also *Goládhyáya*, II, 3, note. The terrestrial days in a *kalpa* are 1577916450000. See *ibid.*, II, 3, note. These two numbers as given in the *Súrya-siddhánta* (I, 37) are slightly different.—Ed.]

fraction $\frac{6}{19}$ to signs (*rásis*), degrees, minutes and seconds. Now as there are 12 signs in one revolution, 30 degrees in one sign, 60 minutes in one degree, and 60 seconds in one minute, we get $\frac{6}{19}$ of a revolution = 3 signs, 23°. 41′.3″$\frac{3}{19}$, and this result indicates the planet's place. Now suppose the remainder of seconds after division by 19, *i.e.*, 3, is alone given, and we have to find the planet's place by an inverse process. Let y denote the remainder of minutes, and x the integral number of seconds. Then it is clear from the process which we adopted in reducing the fraction $\frac{6}{19}$, that $\frac{60y-3}{19}=x$, the general solution of which is given (§256) by $x=60-(57+60t)$, $y=19-(18+19t)$. The only positive integral solution is got by putting $t=0$; then $x=3$, $y=1$. The quotient x is the number of seconds, *viz.*, 3; and the multiplier y is the remainder of minutes, *viz.*, 1. It is easy to see that there can be only one positive integral solution satisfying the conditions of the problem. For x must obviously be less than 60, and y less than 19; so that $57+60t$ must be positive and less than 60, and $18+19t$ must be positive and less than 19. Hence there can be only *one* value of t satisfying these conditions, and consequently only *one* positive integral solution satisfying the problem. Making the necessary changes in the coefficient of y and in the subtractive quantity, and repeating the above process, we clearly obtain the number of minutes, degrees and signs indicating the planet's place, and the elapsed days. Thus the reason for the rule is clear.

Similarly, to find the number of solar days which have elapsed from the beginning of a *kalpa* up to any given epoch, suppose S, y, denote respectively the *saura* days in the *kalpa* and the *saura* days elapsed, and A, a, the corresponding additive months. Now since one additive month occurs in every $32\frac{1}{2}$ solar months (*Golādhyāya*, IV, 9,10), we evidently get the proportion, $S:A::y:a$, whence $a=\frac{Ay}{S}=x+\frac{b}{S}$ suppose, x being an integer, and $\frac{b}{S}$ a proper fraction. Consequently, $\frac{Ay-b}{S}=x$, and a positive integral solution of this equation will give the solar days and the integral number of additive months that have elapsed. Since by supposition, y is less than S, and x less than A, we can show as above that there can be only *one* positive integral solution, satisfying the conditions of the problem.

The case of finding the elapsed lunar days from the given remainder of deficient days or *avamas*, is precisely similar to the above, it being observed that an *avama* occurs in $64\frac{1}{11}$ lunar days (*Goládhyáya*, IV, 12).

In a period of $32\frac{1}{2}$ solar months there are $33\frac{1}{2}$ lunar months very nearly; this excess of the number of lunar months, *viz.*, one lunar month is called an *adhimása* or additive month, because a proportionate multiple of it is to be added to the solar months in any given period in order to convert them into lunar months. Again, in a period of $64\frac{1}{11}$ lunar days there are $63\frac{1}{11}$ terrestrial or mean solar days very nearly; this difference between the two numbers, *viz.*, one mean solar day, is called an *avama* or subtractive day, because a proportionate multiple of it is to be subtracted from the lunar days in any given period in order to convert them into mean solar days.]

265. Rule for a conjunct pulverizer.[1] If the divisor be the same and the multipliers various, then, making the sum of those multipliers the dividend, and the sum of the remainders a single remainder, and applying the foregoing method of investigation, the precise multiplier so found is denominated a conjunct one.

[The reason for the rule will appear from the solution of the example which follows.]

266. Example. What quantity is it, which multiplied by five, and divided by sixty-three, gives a residue of seven; and the same multiplied by ten

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[1] *Sanslishta-kuttaka* or *sanslishtasphuta-kuttaka*, a distinct pulverizing multiplier belonging to conjunct residues.—Gan. A multiplier deduced from the sum of multipliers and that of remainders.—Súr.

and divided by sixty-three, a remainder of fourteen? Declare the number.[1]

Here the sum of the multipliers is made the dividend, and the sum of the residues, a subtractive quantity; and the statement is as follows :—

dividend 15
divisor 63. Subtractive 21. Or reduced to least terms :—

dividend 5
divisor 21. Subtractive 7.

Proceeding as before,[2] the multiplier is found 14.

[In this example we have two simultaneous equations involving three unknown quantities. Let y = quantity required. Then we have evidently, $5y = 63m + 7$, $10y = 63n + 14$, where m and n are certain positive integers. Put $m + n = x$; thus, $63x - 15y = -21$, whence y can be found by §256, and the reason for the rule in §265 is obvious.]

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[1] [See *Goládhyáya*, XIII, 13–15.—Ed.]

[2] The quotient as it comes out in this operation is not to be taken: but it is to be separately sought with the several original multipliers applied to this quantity and divided by the divisor as given.—Gan.

CHAPTER XIII.
COMBINATION OF DIGITS.[1]

267. Rule.[2] The product of multiplication of the arithmetical series beginning and increasing by unity and continued to the number of places, will be the variations of number with specific figures: that divided by the number of digits and multiplied by the sum of the digits, being repeated in the places of figures and added together, will be the sum of the permutations.

[Let there be n digits. Then evidently there are $\underline{|n}$ numbers which can be formed with all these digits. Consider any one of these digits, and denote it by d. In $\underline{|n-1}$ cases, d is in the units' place, in as many cases d is in the tens' place, in as many cases d is in the hundreds' place, and so on. Thus the sum arising from the d alone is $\underline{|n-1}\{d + 10d + 100d + \ldots . 10^{n-1}d\}$. Proceeding similarly with the other digits, we get the sum of all the numbers $= \underline{|n-1} \times$ sum of digits $\times$ $(10^{n-1} + \ldots + 10 + 1)$

$= \dfrac{\underline{|n}}{n} \times$ sum of digits $\times$ $(10^{n-1} + \ldots + 10 + 1)$, which stated in words leads to the rule. The meaning of the phrase, *being repeated in the places of figures and added together*, is obvious. See foot-note to §13.]

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[1] *Anka-pása-vyavahára*, concatenation of digits: a mutual mixing of the numbers, as it were a rope of numerals, their variations being likened to a coil. See Gan. and Súr.

[2] To find the number of the permutations and the sum or amount of them, with specific numbers.—Gan. and Súr.

268. Example. How many variations of number can there be with two and eight, or with three, nine and eight, or with the continued series from two to nine? Tell promptly the several sums of these numbers.

Statement of the first example: 2, 8. Here the number of places is 2. The product of the series from 1 to the number of places and increasing by unity, will be 2. Thus the permutations of number are found 2. That product 2, multiplied by the sum of the figures 10, is 20; and divided by the number of digits 2, is 10. This repeated in the places of figures ($\begin{smallmatrix}10\\10\end{smallmatrix}$) and added together, is 110, the sum of the numbers.

Statement of the second example: 3, 9, 8.

The arithmetical series is 1, 2, 3, of which the product is 6; and so many are the variations of number. That multiplied by the sum 20, is 120; which divided by the number of digits 3, gives 40; and this, repeated in the three places of figures and summed, makes 4440, the sum of the numbers.

Statement of the third example: 2, 3, 4, 5, 6, 7, 8, 9. The arithmetical series beginning and increasing by unity is 1, 2, 3, 4, 5, 6, 7, 8. The product gives the permutation of numbers, 40320. This, multiplied by the sum of the figures 44, is 1774080, which divided by the number of terms 8, is 221760; and the quotient being repeated in the eight places of figures and summed, the total is the sum of the numbers, 24639999975360.

269. Example. How many are the variations of form of the god *Sambhu* by the exchange of his ten attributes held reciprocally in his several hands, namely, the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the

arrow, and the bow[1] as those of *Hari* by the exchange of the mace, the discus, the lotus and the conch?

Statement: number of places 10.

In the same mode, as above shown, the variations of form are found 3628800. So the variations of form of *Hari* are 24.

270. Rule.[2] The permutations found as before, being divided by the permutations separately computed for as many places as are filled by like digits, will be the variations of number, from which the sum of the numbers will be found as before.

[Let there be n digits; and suppose p of them to be d_1, q of them to be d_2, and the rest unlike, namely, d_3, d_4, &c. Then the variations of number will clearly be $= \frac{\lfloor n}{\lfloor p \lfloor q}$. (See Todhunter's Algebra, Art. 497.) The number of cases in which d_1 is in the units' or tens' or hundreds' &c. place is $\frac{\lfloor n-1}{\lfloor p-1 \lfloor q}$ (Todhunter's Algebra, Art. 497); and hence the sum arising from d_1

[1] *Sambhu* or *Siva* is represented with ten arms, and holding in his ten hands the ten weapons or symbols here specified; and, by changing the several attributes from one hand to another, a variation may be effected in the representation of the idol, in the same manner as the image of *Hari* or *Vishnu* is varied by the exchange of his four symbols in his four hands. The twenty-four different representations of *Vishnu*, arising from this diversity in the manner of placing the weapons or attributes in his four hands, are distinguished by as many discriminative titles of the god allotted to those figures in the theogonies or *Purānas*. It does not appear that distinct titles have been in like manner assigned to any of the more than three millions of varied representations of *Siva*.

The ten attributes of *Siva* are :—1st, *pása*, a rope or chain for binding an elephant; 2nd, *ankusa*, a hook for guiding an elephant; 3rd, *ahi*, a serpent; 4th, *damaru*, a tabor; 5th, *kapála*, a human skull; 6th, *trisúla*, a trident; 7th, *khatwánga*, a bedstead, or a club in the form of the foot of one; [it may also mean a club having a skull at the top.—Ed.] 8th, *sakti*, a dagger; 9th, *sara*, an arrow; 10th, *chápa*, a bow.

[2] Special; being applicable when two or more of the digits are alike.

alone is $\frac{|\underline{n-1}}{|\underline{p-1}\,|\underline{q}}(10^{n-1}+\ldots\ldots+10+1)\,d_1$. Similarly the sum arising from d_2 is $\frac{|\underline{n-1}}{|\underline{p}\,|\underline{q-1}}(10^{n-1}+\ldots\ldots+10+1)d_2$, and that arising from d_3, d_4, &c., is

$$\frac{|\underline{n-1}}{|\underline{p}\,|\underline{q}}(10^{n-1}+\ldots\ldots+10+1)\,(d_3+d_4+\&c.)$$

Hence the sum of all the numbers is

$$\frac{|\underline{n}}{n}(10^{n-1}+\ldots\ldots+10+1)\left\{\frac{d_1}{|\underline{p-1}\,|\underline{q}}+\frac{d_2}{|\underline{p}\,|\underline{q-1}}+\frac{d_3+d_4+\&c.}{|\underline{p}\,|\underline{q}}\right\}$$

$$=\frac{|\underline{n}}{n\,|\underline{p}\,|\underline{q}}\;(pd_1+qd_2+d_3+d_4+\&c.)\;(10^{n-1}+\ldots+10+1)$$

$$=\frac{|\underline{n}}{n\,|\underline{p}\,|\underline{q}}\times\text{sum of digits}\times(10^{n-1}+\ldots+10+1),$$ whence the rule.]

271. Example. How many are the numbers with two, two, one and one? And tell me quickly, mathematician, their sum: also with four, eight, five, five and five, if thou be conversant with the rule of permutation of numbers.

Statement of the 1st example: 2, 2, 1, 1. Here the permutations found as before (§267) are 24. First, two places are filled by like digits (2, 2), and the permutations for that number of places are 2. Next two other places are filled by like digits (1, 1), and the permutations for these places are also 2. Total 4. The permutations 24 divided by 4 give 6 for the variations of number: *viz.*,[1] 2211, 2121, 2112, 1212, 1221, 1122. The sum[2] of the numbers is found as before 9999.

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[1] The enumeration of the possible combinations is termed *prastára*.

[2] The variations 6, multiplied by the sum of the figures 6, and divided by the number of digits 4, give 9; which being repeated in four places of figures and summed, makes 9999.

Statement of the 2nd example: 4, 8, 5, 5, 5. Here the permutations found as before are 120, which, divided by the permutations for three places, *viz.*, 6, give the variations 20: *viz.*, 48555, 84555, 54855, 58455, 55485, 55845, 55548, 55584, 45855, 45585, 45558, 85455, 85545, 85554, 54585, 58545, 55458, 55854, 54558, 58554. The sum[1] of the numbers comes out 1199988.

272. Rule[2]: half a stanza. The series of the numbers decreasing by unity from the last[3] to the number of places, being multiplied together, will be the variations of number, with dissimilar digits.

[This rule gives the ordinary formula for the number of permutations of n things taken r at a time, *viz.*, $n\ (n-1)\ (n-2) \ldots\ldots (n-r+1)$.]

273. Example. How many are the variations of number with any digits except cipher exchanged in six places of figures? If thou know, declare them.

The last number is nine. Decreasing by unity, for as many as are the places of figures, the statement of the series is 9. 8. 7. 6. 5. 4. The product of these is 60480.

274. Rule[4]: two stanzas. If the sum of the digits be determinate, the arithmetical series of numbers from one less than the sum of the digits, decreasing by unity, and continued to one less than the places, being divided by one and so forth, and the quotients being

[1] The variations 20, multiplied by the sum of the figures 27, give 540, which, divided by the number of digits 5, makes 108: and this being repeated in five places of figures and summed, yields 1199988.

[2] To find the variations for a definite number of places with indeterminate digits.—Gan.

[3] That is, from nine [in the example which follows.]—Gan.

[4] To find the permutations with indeterminate digits for a definite sum and a specific number of places.—Gan.

multiplied together, the product will be equal to the variations of number. This rule must be understood to hold good, provided the sum of the digits be less than the number of places added to nine.

A compendium only has been here delivered for fear of prolixity, since the ocean of calculation has no bounds.

[Let $s =$ the sum of the digits,
$n =$ the number of digits,
and let $s = n + m$.

Then by supposition, $n + m < n + 9$, or $m < 9$, or $m + 1$ not > 9, so that even if $n - 1$ of the n digits be 1's, the remainder of the sum, $m + 1$ being not > 9, can form the remaining digit.

Now let the n 1's composing n be denoted by 1^n, 1^{n-1}, 1^{n-2} ....1^1, and the m 1's composing m, by 1_1, 1_2, 1_3......1_m. Then, if we fix 1^n in the first place on the left, and take the different permutations of the remaining $n - 1 + m$ symbols 1^{n-1}, 1^{n-2},......1^1 and 1_1, 1_2, 1_3......1_m, of which the $n - 1$ indexed 1's are considered to be alike and of one sort, and the other m 1's are considered to be alike and of another sort, and place each of these permutations to the right of 1^n, and regard the sum of each indexed 1 with the group of 1's with suffixes, if any, following it on its right as forming a digit of one of the required numbers,

we shall have a series of numbers like the following :—

$(1^n\ 1_1\ 1_2)\ (1^{n-1}\ 1_3\ 1_4\ 1_5) \ldots (1^p \ldots 1_{m-1}) \ldots (1^2\ 1_m)\ (1^1)$,

$(1^n\ 1_1\ 1_2\ 1_3\ 1_4)\ (1^{n-1}) \ldots (1^p \ldots 1_n) \ldots (1^2)\ (1^1)$,

....

This series will evidently contain all the required numbers and those alone ; and the number of these numbers being the number required, the problem is reduced to finding the number of permutations of $n + m - 1$ things taken all together, of which $n - 1$ are alike and of one sort, and m are alike and of another sort.

And this number $= \dfrac{(n+m-1)!}{(n-1)!\ m!}$

$$= \frac{(n+m-1)\ (n+m-1-1) \ldots (n+m-1-\overline{n-1})\ (m-1) \ldots 1}{(n-1)!\ m!}$$

$$= \frac{(n+m-1)\ (n+m-2)\ \ldots\ (n+m-1-\overline{n-2})}{(n-1)!}$$

$$= \frac{(s-1)\ (s-2)\ \ldots\ (s-\overline{n-1})}{1\ .\ 2 \ldots\ldots\ldots (n-1)},$$

which proves the rule.]

275. Example. How many various numbers are there, with digits standing in five places, the sum of which is thirteen ? If thou know, declare them.

Here the sum of the digits less one is 12. The decreasing series from this to one less than the number of digits, divided by unity, &c. being exhibited, the statement is, $\frac{12}{1}$. $\frac{11}{2}$. $\frac{10}{3}$. $\frac{9}{4}$. The product of their multiplication ($\frac{11880}{24}$) is equal to the variations of the number[1], 495.

276. Though neither multiplier nor divisor be asked, nor square, nor cube, still presumptuous inexpert scholars in arithmetic will assuredly fail in (problems on) this combination of numbers.

277. Joy and happiness is indeed ever increasing in this world for those who have *Lílávatí* clasped to their throats[2], decorated as the members are with neat reduc-

[1] 91111, 52222, 13333, each five ways; 55111, 22333, each ten ways; 82111, 73111, 64111, 43222, 61222, each twenty ways; 72211, 53311, 44221, 44311, each thirty ways; 63211, 54211, 53221, 43321, each sixty ways. Thus the total is 495.

[2] By constant repetition of the text. This stanza, ambiguously expressed and bearing a double import, implies a simile : as a charming woman closely embraced, whose person is embellished by an assemblage of elegant qualities, who is pure and perfect in her conduct, and who utters agreeable discourse. See Gan.

tion of fractions, multiplication and involution, pure and perfect as are the solutions, and tasteful as is the speech which is exemplified.

[In Pandit Jívánanda Vidyáságara's edition of the original, there is a stanza after the above, showing the varied scholarship of Bháskara, and stating that the present work was composed by him. It was probably added by some pupil of Bháskara, and so it has been omitted by Colebrooke. It runs as follows:— "The author of this (*Lílávatí*) is that illustrious Bháskara, (a scholar) of vast erudition, who thoroughly mastered eight works on grammar, (*viz.*, those of Indra, Chandra, Kásakritsni, Ápisali, Sákatáyana, Pánini, Amara, and Jainendra),[1] six works on medical science, (*viz.*, *Agnivesa-sanhitá*, *Bheda-sanhitá*, *Játúkarna-sanhitá*, *Parásara-sanhitá*, *Sírapáni-sanhitá*, and *Hárita-sanhitá*),[2] the six philosophical systems (*viz.*, *Sánkhya*, *Yoga*, *Nyáya*, *Vaiseshika*, *Mímánsá* and *Vedánta*), five works on *ganita* (calculation), (*viz.*, *Paulisa-siddhánta*, *Romaka-siddhánta*, *Básishtha-siddhánta*, *Súrya-siddhánta* and *Paitámaha-siddhánta*),[3] and the four *Vedas* (*viz.*, the *Rik*, the *Yajush*, the *Sáman* and the *Atharvan*); and who understood the three *Ratnas*, (*i.e.*, the three *Prasthánas* of the *Vedánta*, *viz.*, the *Sútras*, the *Upanishads* and the *Prakaranas*), as well as the two *Mímánsás*,[4] and the one eternal *Brahman*, the aim and scope of both."]

[[1] See Bibliotheca Indica, Nirukta, Vol. IV, Appendix, page *jau*. Some of these authors composed dictionaries and not works on grammar. Thus the word *vyákaranáni* in the original has been rather loosely used.—ED.]

[[2] These six ancient works form the basis of the later works of Charaka, Susruta and Bágbhata. The works of Charaka and Susruta are usually called *sanhitás*; that of Bágbhata is known under the name of *Ashtánga-hridaya*.—ED.]

[[3] See Varáha-mihira's *Vrihat-sanhitá*, Ch. II.—ED.]

[[4] Namely, the *Púrva-mímánsá* of Jaimini, usually called the *Mímánsá*, and the *Uttara-mímánsá* of Vyása, usually called the *Vedánta*.—ED.]